1970

A Tauberian Theorem For Borel-type Methods Of Summability

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A TAUBERIAN THEOREM FOR BOREL-TYPE METHODS OF SUMMABILITY

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Submitted in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Faculty of Graduate Studies

The University of Western Ontario

London, Canada.

September, 1969

ABSTRACT

A series of real or complex numbers $\sum_{n=o}^{\infty} a_n$ is said to be *summable* (B,α,β) to A if

$$\alpha e^{-x} \sum_{n=N}^{\infty} A_n \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} \rightarrow A \qquad \text{as} \quad x \rightarrow \infty$$

where $\alpha > 0$, β is real, and N is a non-negative integer such that $\alpha N + \beta > 0$. The standard Borel exponential method of summability B is just the special case (B,1,1).

The main result of this thesis is the following "O" Tauberian Theorem for summability (B,α,β):

If $a_n = O(n^{-1/2})$ (<u>i. e.</u> *the sequence* $\{\sqrt{n}\, a_n : n = 1,2,3,\ldots\}$ *is bounded*) , *and* $\sum_{n=o}^{\infty} a_n = A\ (B,\alpha,\beta)$, *then* $\sum_{n=o}^{\infty} a_n = A$.

A known method of summability (e,c) is considered and it is shown that summability (B,α,β) is equivalent to summability $(e,\alpha/2)$ if $a_n = o(1)$. It is then shown that if $0 < d < c$, summability (e,c) and $a_n = o(1)$

together imply summability (e,d). This is done by considering another known method of summability (γ,k), showing that summability (γ,k) implies summability (γ,ℓ) if $0 < \ell < k$, and that, if $a_n = o(1)$, summability (γ,k) is equivalent to summability (e,c) where $c = \frac{k}{2(1-k)}$. Summability (B,α,β) and $a_n = O(n^{-1/2})$ are shown to imply that $A_n = O(1)$. Vitali's Theorem is then used to show that if $A_n = O(1)$, summability (B,α,β) implies summability (e,c) for all positive c. The desired "O" Tauberian Theorem is then deduced.

ACKNOWLEDGEMENTS

I am deeply indebted to Dr. D. Borwein for his kindness, encouragement and academic guidance.

I would also like to express my thanks to my wife who typed the first draft of this thesis, and to Miss Christine McCulloch who typed the final draft.

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INTRODUCTION

(1) Notation.

Let $\sum_{n=0}^{\infty} a_n$ be a series of complex (or real) numbers. Let A_n denote the partial sum $a_0 + \ldots + a_n$ of the series if $n \geq 0$ and let $A_n = 0$ if $n < 0$.

The symbol [x] denotes the greatest integer $\leq x$.

(2) Definitions and discussion.

If a given method of summability T assigns the "sum" A to the series $\sum_{n=0}^{\infty} a_n$, we say that $\sum_{n=0}^{\infty} a_n$ is *summable* T to A and write $\sum_{n=0}^{\infty} a_n = A$ (T) or $A_n \to A$ (T).

The method T is said to be *regular* if it sums every convergent series to its ordinary sum.

Now suppose that $\alpha > 0$, β is real, and N is a non-negative integer such that $\alpha N + \beta > 0$. We say that the series $\sum_{n=0}^{\infty} a_n$ is summable (B,α,β) to A and write

$$\sum_{n=0}^{\infty} a_n = A(B,\alpha,\beta) \text{ if } \alpha e^{-x} \sum_{n=N}^{\infty} \frac{A_n x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} \to A \text{ as } x \to \infty.$$

Since $e^{-x} \dfrac{A_k \; x^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)} \to 0$ as $x \to \infty$, it is clear that the actual choice of N is immaterial.

The method (B,1,1) is the standard Borel exponential method of summability B.

For the definition and early investigation of summability (B,α,β) see Borwein [2], [3], and [4], Good [6], Hardy [7], and Włodarski [14]. For Borel summability B see Hardy [8].

The method (B,α,β) is equivalent to a special case of a very general type of summability method, usually known as a J method or integral function method (although entire function method would perhaps be more apt). Let

$J(z) = \sum_{n=o}^{\infty} j_n z^n$, $j_n \geq 0$, where J(z) is an entire function (*i.e.* analytic in the entire complex plane) but not a polynomial. Then $\sum_{n=o}^{\infty} a_n = A\ (J)$ if

$$\frac{\sum_{n=o}^{\infty} j_n A_n x^n}{J(x)} \to A \quad \text{as } x \to \infty.$$

Since $\sum_{n=N}^{\infty} \dfrac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} \sim \dfrac{e^x}{\alpha}$ as $x \to \infty$ (see Borwein [4], p. 130, with $\delta = \beta-1$), it follows that the (B,α,β) method is equivalent to the J method defined by setting

$$j_n = \begin{cases} \dfrac{1}{\Gamma(\alpha n+\beta)} & , \quad n \geq N \\ 0 & , \quad 0 \leq n < N. \end{cases}$$

(Substituting y for x^α and then writing x for y gives

$\dfrac{1}{J(x)} \sim \alpha x^{(\beta-1)/\alpha} e^{-x^{1/\alpha}}$. Then setting $u = x^{1/\alpha}$, it follows

that $\dfrac{\sum_{n=o}^{\infty} j_n A_n x^n}{J(x)} \sim \alpha e^{-u} \sum_{n=N}^{\infty} \dfrac{A_n u^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}$.)

Since J methods are known to be regular (see Hardy [8], Theorem 33]), summability (B,α,β) is a regular method.

The main object in this thesis is to prove the following theorem.

If $\sum_{n=o}^{\infty} a_n = A \ (B,\alpha,\beta)$ *and* $a_n = O(n^{-1/2})$ *then* $\sum_{n=o}^{\infty} a_n = A$.

(This is Theorem (6.7) of the thesis; see the end of this introduction for the method of numbering.)

This is an example of a "O" Tauberian Theorem in that we have the summability of a series by a regular method together with a "O" condition (the Tauberian condition) implying its convergence.

The corresponding theorem for Borel summability B was first proved by Hardy and Littlewood [9]. It is the main result of chapter nine of Hardy's book [8].

That summability (B,α,β) of $\sum_{n=o}^{\infty} a_n$ and $a_n = o(n^{-1/2})$

together imply the convergence of the series is a special case of Borwein [5]. (This is the "o" Tauberian Theorem for summability (B,α,β).)

Borwein [4, (III) and (IV)] has proved that:

If $J(z) = \sum_{n=N}^{\infty} \frac{z^n}{h(n)}$ *where* $h(z)$ *is an analytic function of* $z = x + iy$ *in the region* $x > x_o$, *such that*

(i) *when* $x > x_o$ *and* $|z|$ *is large*

$$h(z) = z^{\alpha z+\beta}\, e^{\gamma z} \left\{C + O\left(\frac{1}{|z|}\right)\right\} \text{ where } C > 0,\ \alpha > 0,$$

β *and* γ *are real,*

and

(ii) $h(x)$ *is real for* $x > x_o$,

then

$$\sum_{n=o}^{\infty} a_n = A\ (J) \text{ if and only if } \sum_{n=o}^{\infty} a_n = A\ (B,\alpha,\beta + 1/2).$$

In particular, taking $h(z) = \{\Gamma(az + b)\}^c\ (z+p)^{qz+r}$ *where* $b,c,p,q,$ *and* r *are real,* $a > 0$ *and* $ac + q > 0$, *so that*

$$J(z) = \sum_{n=N}^{\infty} \frac{z^n}{\{\Gamma(an+b)\}^c\ (n+p)^{qn+r}}$$

$$\sum_{n=o}^{\infty} a_n = A\ (J) \text{ if and only if}$$

$$\sum_{n=o}^{\infty} a_n = A\ (B,\ ac+q,\ bc + r - c/2 + 1/2).$$

It follows that Theorem (6.7) is in fact a Tauberian Theorem for quite a wide class of summability methods.

Theorem (6.7) remains true if $\sum_{n=0}^{\infty} a_n = A\ (B,\alpha,\beta)$ is replaced by $\sum_{n=0}^{\infty} a_n = A\ (B',\alpha,\beta)$, by which it is meant that, as $y \to \infty$,

$$\int_0^y e^{-x}\, dx \sum_{n=N}^{\infty} \frac{a_n x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} \to A - A_{N-1}\ .$$

This is a consequence of the following known result (see Borwein [3, Theorem 2]):

$$\sum_{n=0}^{\infty} a_n = A\ (B,\alpha,\beta+1) \textit{ if and only if } \sum_{n=0}^{\infty} a_n = A\ (B',\alpha,\beta).$$

(3) Method of numbering.

The thesis contains this introduction and six chapters. Theorems, lemmas, corollaries, and equations to which we will later refer are numbered consecutively within each chapter. The number is placed first and is in the margin for ease of reference. Thus in chapter three, our first result is listed as (3.1) THEOREM and would be referred to as (3.1) or Theorem (3.1).

CHAPTER 1

KNOWN RESULTS AND SOME PRELIMINARY LEMMAS

We first state some results of D. Borwein [5] which are needed in later chapters.

(1.1) *If* $\rho \geq -1/2$, $a_n = o(n^\rho)$ *and* $\sum_{n=o}^{\infty} a_n = 0 \ (B,\alpha,\beta)$,

then $A_n = o(n^{\rho+1/2})$.

This is Lemma 5 of Borwein [5] with $k = 0$, $\mu = 1$, and $\lambda = \rho$. The result we need follows immediately:

(1.2) *If* $\rho \geq -1/2$, $a_n = o(n^\rho)$ *and* $\sum_{n=o}^{\infty} a_n = A \ (B,\alpha,\beta)$,

then $A_n = o(n^{\rho+1/2})$.

Let $x > 0$, $h = n - x/\alpha$, $1/2 < \zeta < 2/3$, and $0 < \eta < 2\zeta-1$.

Then

(1.3) $$e^{-x} \sum_{|h|>x^\zeta} \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} = O\left(e^{-x^\eta}\right)$$

(Lemma 2, part (d) of Borwein [5])

and

$$(1.4)\quad e^{-x}\,\frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} = \frac{1}{\sqrt{2\pi x}}\; e^{-\alpha^2 h^2/2x}\;\{1 + O(x^{3\zeta-2})\}$$

if $|h| \le x^{\zeta}$

(Lemma 2, part (e) of Borwein [5]).

In fact we need the slightly sharper estimate:

$$(1.5)\quad e^{-x}\,\frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} = \frac{1}{\sqrt{2\pi x}}\; e^{-\alpha^2 h^2/2x}\;\left\{1 + O\left(\frac{|h|+1}{x}\right) + O\left(\frac{|h|^3}{x^2}\right)\right\}$$

if $|h| \le x^{\zeta}$.

This would have been obtained if Borwein had not used $\frac{|h|+1}{x} = O(x^{3\zeta-2})$ and $\frac{|h|^3}{x^2} = O(x^{3\zeta-2})$ in simplifying near the end of his proof. In (1.5) we write $O\left(\frac{|h|+1}{x}\right)$ instead of $O\left(\frac{|h|}{x}\right)$ in order to include the case $h = 0$.

It follows from Stirling's Theorem that

$$(1.6)\quad \Gamma(\alpha n+\beta) = (2\pi)^{1/2}\, e^{\alpha n}\, (\alpha n)^{\alpha n+\beta-1/2}\;\{1 + O(1/n)\}$$

(See *Higher Transcendental Functions* [7, formula 3, 1.18, p. 47, vol. 1].)

It follows that $\frac{(\alpha n)^{1/2}\,\Gamma(\alpha n+\beta-1/2)}{\Gamma(\alpha n+\beta)} \sim 1$ and hence that

$$(1.7)\quad \frac{n^{1/2}}{\Gamma(\alpha n+\beta)} = O\left(\frac{1}{\Gamma(\alpha n+\beta-1/2)}\right).$$

The next four well known results are stated without proof and will be used without reference.

$x^{\lambda}e^{-px^{a}} = o(1)$ as $x \to \infty$ for all real λ, $p > 0$, $a > 0$.

For $n \geq 0$:

$$n^{\lambda}e^{-\mu n} = 0\,(e^{-\gamma n}) \text{ if } 0 < \gamma < \mu\ ,\ \lambda \text{ real.}$$

$$e^{-\mu n} = 0\left(e^{-n^{\eta}}\right) \quad \text{if } \mu > 0\ ,\ \eta > 0\ .$$

$$e^{-cn^{a}} = 0\left(e^{-n^{b}}\right) \quad \text{if } c > 0 \text{ and } 0 < b < a\ .$$

(1.8) LEMMA.

For $0 < u \leq H$, $e^{0(u)} = 1 + 0(u)$.

PROOF.

Let $v = 0(u)$ where $0 < u \leq H$. Then $|v| \leq Ku$ for some constant K, and so

$$|e^{v} - 1| = \left|v\sum_{n=1}^{\infty}\frac{v^{n-1}}{n!}\right| \leq u\left\{K\sum_{n=1}^{\infty}\frac{(KH)^{n-1}}{n!}\right\} = Cu$$

for some constant C, since the series converges. The result follows.

(1.9) LEMMA.

For $|u| \leq r < 1$, $\log\,(1 + u) = u - u^{2}/2 + 0(|u|^{3})$.

PROOF.

For $|u| < 1$,

$$\log (1+u) = \sum_{n=1}^{\infty} (-1)^{n-1} u^n/n = u - u^2/2 + u^3 \sum_{n=o}^{\infty} (-1)^n \frac{u^n}{n+3} ,$$

and therefore

$$|\log (1+u) - (u - u^2/2)| \leq |u|^3 \sum_{n=o}^{\infty} \frac{|u|^n}{n+3}$$

$$\leq |u|^3 \sum_{n=o}^{\infty} \frac{r^n}{n+3} = C |u|^3 \quad \text{for some constant } C ,$$

since the series converges. The result follows.

In the rest of this chapter some order relations involving partial sums of the series $\sum_{n=o}^{\infty} a_n$, which will be needed later, are stated and proved as lemmas.

We assume throughout that $1/2 < \zeta < 2/3$, $0 < k < 1$, and n is a positive integer.

(1.10) LEMMA.

If $\lambda > 0$ *and* $A_n = o(n^\lambda)$ *then* $A_{j+h} = 0 \left[(j+|h|)^\lambda\right]$ *where* j *is a positive integer. In particular this holds for either* $j = n$ *or* $j = M = [n/k]$.

PROOF.

Since $A_n = o(n^\lambda)$ there exists an N_o such that if $j + h > N_o$ then $|A_{j+h}| \leq (j+h)^\lambda \leq (j+|h|)^\lambda$. Since there is only a finite number of integers between 0 and N_o, there is a constant $K > 0$ such that $|A_{j+h}| \leq K(j+|h|)^\lambda$ for $j+h = 0,1,2,\ldots,N_o$. Thus $A_{j+h} = O\,(j+|h|)^\lambda$.

(1.11) LEMMA.

If $a_n = o(1)$ *and* $|h| \leq n^\zeta$*, then,* *for either* $j = n$ *or* $j = M = [n/k]$,
$A_{j+h} - A_j = o(|h|)$ *as* $n \to \infty$, *uniformly for* $|h| \leq n^\zeta$.
(This is a special case of Hardy [8, Theorem 144] with $\alpha = -1$, $\beta = 1$, $\rho = 0$ since for n sufficiently large $n^\zeta \leq n/2 = Hn$. Our case is easy and we prove it directly.)

PROOF.

Since $a_n = o(1)$, given $\varepsilon > 0$ there exists an N_o such that if $r \geq N_o$ then $|a_r| < \varepsilon$. Thus

$$|A_{j+h} - A_j| \leq \left\{ \sup_{j-n^\zeta \leq r \leq j+n^\zeta} |a_r| \right\} |h| \leq \varepsilon\ |h|$$

if $j - n^\zeta \geq N_o$.

Since for either $j = n$ or $j = M$, $j - n^\zeta \to \infty$ as $n \to \infty$, it follows that $A_{j+h} - A_j = o(|h|)$ as $n \to \infty$, uniformly for $|h| \leq n^\zeta$.

(1.12) LEMMA.

If $A_n = o(n^{1/2})$ *and* $|h| \le (\alpha n)^{\zeta}$, *then,*

for either $j = n$ *or* $j = M = [n/k]$,

$A_{j+h} = o(j^{1/2})$ *as* $n \to \infty$, *uniformly for* $|h| \le (\alpha n)^{\zeta}$.

PROOF.

$$\frac{|A_{j+h}|}{j^{1/2}} = \left(1 + h/j\right)^{1/2} \frac{|A_{j+h}|}{(j+h)^{1/2}}$$

$$\le \left(1 + \frac{(\alpha n)^{\zeta}}{j}\right)^{1/2} \frac{|A_{j+h}|}{(j+h)^{1/2}}$$

$$\le H \frac{|A_{j+h}|}{(j+h)^{1/2}} \quad \text{for some constant } H > 0,$$

since $\left(1 + \frac{(\alpha n)^{\zeta}}{j}\right)^{1/2} \to 1$ as $n \to \infty$ for either $j = n$ or $j = [n/k]$. Because $|h| \le (\alpha n)^{\zeta}$, $j + h \ge j - (\alpha n)^{\zeta}$. Since for either $j = n$ or $j = M$, $j - (\alpha n)^{\zeta} \to \infty$ as $n \to \infty$,

$$\frac{|A_{j+h}|}{(j+h)^{1/2}} = o(1) \text{ as } n \to \infty, \text{ uniformly for } |h| \le (\alpha n)^{\zeta}.$$

Therefore $A_{j+h} = o(j^{1/2})$ as $n \to \infty$, uniformly for

$|h| \le (\alpha n)^{\zeta}$ $\quad (j = n$ or $j = M)$.

(1.13) LEMMA.

If $A_n = o(n^{1/2})$ *and* $|h| > (\alpha n)^\zeta > 0$, *then, for either* $j = n$ *or* $j = M = [n/k]$, $A_{j+h} = O(|h|)$.

PROOF.

It follows from Lemma (1.10) with $\lambda = 1/2$ that $A_{M+h} = O\{(M+|h|)^{1/2}\}$. Since $M = [n/k] = O(n)$, $M+|h| = O(n+|h|)$ and $(M+|h|)^{1/2} = O\{(n+|h|)^{1/2}\}$. Thus in either case $A_{j+h} = O\{(n+|h|)^{1/2}\}$.

Now $1/2 < \zeta < 2/3$ and therefore $3/2 < 1/\zeta < 2$ and $3/4 < 1/2\zeta < 1$.

Since $|h| > (\alpha n)^\zeta$, $n = O(|h|^{1/\zeta})$.

Thus

$$n + |h| = O(|h|^{1/\zeta}) + |h| = O(|h|^{1/\zeta})$$

and

$$(n + |h|)^{1/2} = O(|h|^{1/2\zeta}) = O(|h|).$$

Therefore $A_{j+h} = O(|h|)$.

(1.14) LEMMA.

If $a_n = O(n^{-1/2})$ *then, for* $|h| \leq n^\zeta$,

$$A_{n+h} - A_n = O\left(\frac{|h|}{\sqrt{n}}\right) .$$

PROOF.

If $h > 0$,

$$|A_{n+h} - A_n| \leq |a_{n+1}| + \ldots + |a_{n+h}|$$

$$\leq H\left(\frac{1}{\sqrt{n+1}} + \ldots + \frac{1}{\sqrt{n+h}}\right) < \frac{Hh}{\sqrt{n}} .$$

If $h < 0$,

$$|A_{n+h} - A_n| \leq |a_{n+h+1}| + \ldots + |a_n|$$

$$\leq H\left(\frac{1}{\sqrt{n+h+1}} + \ldots + \frac{1}{\sqrt{n}}\right) \leq \frac{H|h|}{\sqrt{n-|h|}}$$

$$\leq \frac{H}{(1-n^{\zeta-1})^{1/2}} \left(\frac{|h|}{\sqrt{n}} \right) .$$

Since $(1 - n^{\zeta-1})^{1/2} \to 1$ as $n \to \infty$ it follows that

$|A_{n+h} - A_n| \leq K \dfrac{|h|}{\sqrt{n}}$ for some constant $K > 0$.

Therefore $|A_{n+h} - A_n| = O\left(\dfrac{|h|}{\sqrt{n}}\right)$.

CHAPTER 2

SOME RELATIONS BETWEEN CERTAIN SUMS AND INTEGRALS

Our main concern in this chapter will be to establish estimates for certain types of sums in terms of integrals. These estimates will enable us, in later chapters, to replace (whenever it is convenient to do so) such a sum by the corresponding integral.

In our first two lemmas we evaluate the integrals which concern us. Throughout the chapter $c > 0$ and n is a positive integer.

(2.1) LEMMA.

$$\int_0^\infty e^{-ct^2/n}\, t^\lambda dt = \begin{cases} (1/2)(\pi n/c)^{1/2} & , \lambda = 0 \\ n/2c & , \lambda = 1 \\ n^2/2c^2 & , \lambda = 3 \end{cases} .$$

PROOF.

Setting $y = (c/n)t^2$, $t = (n/c)^{1/2} y^{1/2}$, $dt = (n/c)^{1/2}(1/2)y^{-1/2}\, dy$, the integral becomes

$$\frac{1}{2}(n/c)^{\frac{\lambda+1}{2}} \int_0^\infty e^{-y} y^{\frac{\lambda-1}{2}} dy = \frac{1}{2}(n/c)^{\frac{\lambda+1}{2}} \Gamma\left(\frac{\lambda+1}{2}\right) .$$

The result follows since $\Gamma(1/2) = \sqrt{\pi}$ while $\Gamma(1) = \Gamma(2) = 1$.

(2.2) LEMMA.

$$\int_{(\alpha n)^{\zeta}}^{\infty} e^{-ct^2/n}\, t\,dt = (n/2c)\, e^{-c\alpha^{2\zeta} n^{2\zeta-1}} \quad ,1/2 < \zeta < 2/3 .$$

PROOF.

Set $u^2 = ct^2/n$, $t = (n/c)^{1/2} u$, $dt = (n/c)^{1/2} du$, and the integral becomes

$$\int_{c^{1/2}\alpha^{\zeta} n^{\zeta-1/2}}^{\infty} e^{-u^2}\, udu = \left(- \frac{e^{-u^2}}{2}\right)_{c^{1/2}\alpha^{\zeta} n^{\zeta-1/2}}^{\infty}$$

$$= (n/2c)\, e^{-c\alpha^{2\zeta} n^{2\zeta-1}} .$$

In the rest of this chapter trivial and rather redundant corollaries of some of the theorems are stated explicitly for ease of later reference.

(2.3) LEMMA.

$$\left| \sum_{h=-\infty}^{\infty} e^{-ch^2/n} - 2\int_0^{\infty} e^{-ct^2/n}\, dt \right|$$

$$= \left| \sum_{h=-\infty}^{\infty} e^{-ch^2/n} - (\pi n/c)^{1/2} \right| < 1 .$$

PROOF.

Let $S = \sum_{h=1}^{\infty} e^{-ch^2/n}$ so that

$\sum_{h=-\infty}^{\infty} e^{-ch^2/n} = 1 + 2S$. Let $f(t) = e^{-ct^2/n}$.

Then since $f'(t) = -(2ct/n)e^{-ct^2/n}$, it is easily seen that f takes its maximum value of 1 at t = 0 and is monotone decreasing (strictly) for $t > 0$.

Letting

$$a(t) = e^{-ch^2/n} \quad , \quad h \leq t < h + 1 ,$$

$$b(t) = e^{-c(h+1)^2/n} \quad , \quad h \leq t < h + 1 ,$$

we have

$$b(t) < f(t) \leq a(t) \quad , \quad 0 \leq t < \infty.$$

Thus

$$S = \int_0^\infty b(t)dt < \int_0^\infty f(t)dt < \int_0^\infty a(t)dt = 1 + S .$$

Since $\int_0^\infty f(t)dt = (1/2)(\pi n/c)^{1/2}$ by (2.1), we have

$$S < (1/2)(\pi n/c)^{1/2} < S + 1.$$

Thus

$(\pi n/c)^{1/2} - 1 < 2S + 1 < (\pi n/c)^{1/2} + 1.$ The theorem follows.

The following corollaries are immediate.

(2.4) COROLLARY.

$$\sqrt{c/\pi n} \sum_{h=-\infty}^{\infty} e^{-ch^2/n} = 1 + O(n^{-1/2}),$$ *uniformly in any finite interval* $0 \leq c \leq K$.

(2.5) COROLLARY.

$$\sum_{h=-\infty}^{\infty} e^{-ch^2/n} = O\left(\int_0^{\infty} e^{-ct^2/n}\, dt \right) .$$

(2.6) LEMMA.

$$\sum_{h=-\infty}^{\infty} |h|\ e^{-ch^2/n} \geq 2 \int_0^{\infty} e^{-ct^2/n}\, tdt - (2n/ec)^{1/2}$$

$$= (n/c) - (2n/ec)^{1/2}$$

For n *sufficiently large,*

$$\sum_{h=-\infty}^{\infty} |h|\ e^{-ch^2/n} \leq 2 \int_0^{\infty} e^{-ct^2/n}\, tdt + (2n/ec)^{1/2}$$

$$= (n/c) + (2n/ec)^{1/2} .$$

PROOF.

Let $S = \sum_{h=1}^{\infty} h\ e^{-ch^2/n}$ so that $\sum_{h=-\infty}^{\infty} |h|\ e^{-ch^2/n} = 2S$.

Let $f(t) = t\ e^{-ct^2/n}$.

Then

$$f'(t) = e^{-ct^2/n} \left(1 - \frac{2ct^2}{n} \right)$$

and

$$f''(t) = \frac{2ct}{n}\, e^{-ct^2/n} \left(\frac{2ct^2}{n} - 3 \right) .$$

It is easily verified that f is monotone increasing for $0 \le t \le (n/2c)^{1/2}$, monotone decreasing for $t \ge (n/2c)^{1/2}$, and takes a maximum value of $(n/2ec)^{1/2}$ when $t = (n/2c)^{1/2}$. Hence f is concave downward for $0 \le t \le (3n/2c)^{1/2}$. Choose an integer h_o such that $h_o - 1 < (n/2c)^{1/2} \le h_o$. Since

$$\sqrt{3}\left(\frac{n}{2c}\right)^{1/2} \ge \left(\frac{n}{2c}\right)^{1/2} + 1 > h_o \text{ if } n > (2+\sqrt{3})c ,$$

it follows that f is concave downward for $h_o - 1 \le t \le h_o$ when $n > (2+\sqrt{3})c$.

Set
$$a(t) = h\, e^{-ch^2/n} , \quad h \le t < h+1 ,$$
$$b(t) = (h+1)\, e^{-\frac{c(h+1)^2}{n}} , \quad h \le t < h+1 .$$

Then
$$b(t) < f(t) \le a(t) \quad \text{for} \quad t \ge h_o ,$$
$$a(t) \le f(t) < b(t) \quad \text{for } 0 \le t \le h_o - 1 .$$

Now
$$\int_0^{h_o} \big(b(t) - a(t)\big)dt = \sum_{h=1}^{h_o} h\, e^{-ch^2/n} - \sum_{h=o}^{h_o - 1} h\, e^{-ch^2/n}$$
$$= h_o\, e^{-ch_o^2/n} = f(h_o)$$

Thus

$$\int_0^\infty f(t)dt \le \int_0^{h_o-1} b(t)dt + \int_{h_o-1}^{h_o} f(t)dt + \int_{h_o}^{\infty} a(t)dt$$

$$= \int_0^\infty a(t)dt + \int_0^{h_o} \big(b(t)-a(t)\big)\,dt + \int_{h_o-1}^{h_o} \big(f(t)-b(t)\big)dt$$

$$\le S + h_o\, e^{-ch_o^2/n} + \left[(n/2ec)^{1/2} - h_o\, e^{-ch_o^2/n}\right]$$

$$= S + (n/2ec)^{1/2} .$$

Thus

$$\sum_{h=-\infty}^{\infty} |h|\, e^{-ch^2/n} \ge \int_0^\infty 2\, e^{-ct^2/n}\, tdt - (2n/ec)^{1/2}$$

$$= (n/c) - (2n/ec)^{1/2} .$$

Also

$$\int_0^\infty f(t)dt \ge \int_0^{h_o-1} a(t)dt + \int_{h_o-1}^{h_o} f(t)dt + \int_{h_o}^{\infty} b(t)dt$$

$$= \int_0^\infty b(t)dt - \int_0^{h_o} \big(b(t)-a(t)\big)dt - \int_{h_o-1}^{h_o} a(t)dt + \int_{h_o-1}^{h_o} f(t)dt$$

$$= S - f(h_o) - f(h_o - 1) + \int_{h_o-1}^{h_o} f(t)dt$$

$$= S - \int_{h_o-1}^{h_o} f(t)dt + 2\left(\int_{h_o-1}^{h_o} f(t)dt - \frac{f(h_o-1)+f(h_o)}{2}\right)$$

$$\geq S - \int_{h_o-1}^{h_o} f(t)dt \quad \text{if} \quad n > (2+\sqrt{3})c \quad , \quad \text{since } f \text{ is then}$$

concave downward in $h_o-1 \leq t \leq h_o$,

$$\geq S - (n/2ec)^{1/2} .$$

Thus for n sufficiently large (certainly if $n > (2+\sqrt{3})c$)

$$\sum_{h=-\infty}^{\infty} |h|\ e^{-ch^2/n} \leq 2 \int_0^{\infty} e^{-ct^2/n}\ tdt + (2n/ec)^{1/2}$$

$$=(n/c)+ (2n/ec)^{1/2} .$$

The following corollary is immediate.

(2.7) COROLLARY.

$$\sum_{h=-\infty}^{\infty} |h|\ e^{-ch^2/n} = 0\left(\int_0^{\infty} e^{-ct^2/n}\ tdt\right) .$$

(2.8) LEMMA.

For $1/2 < \zeta < 2/3$,

$$\sum_{|h|>(\alpha n)^\zeta} |h|\ e^{-ch^2/n} = 0\left(\int_{(\alpha n)^\zeta}^{\infty} e^{-ct^2/n}\ tdt\right) .$$

PROOF.

Since $\zeta > 1/2$, $(\alpha n)^\zeta > (n/2c)^{1/2} + 1$ for n sufficiently large, and thus, with the notation used in the proof of Theorem (2.6)

$b(t) < f(t)$ for $t \geq (\alpha n)^\zeta$.

Then

$$\int_{(\alpha n)^\zeta}^{\infty} b(t)dt \le \int_{(\alpha n)^\zeta}^{\infty} e^{-ct^2/n}\, tdt .$$

That is

$$\sum_{h>(\alpha n)^\zeta} h\, e^{-ch^2/n} \le \int_{(\alpha n)^\zeta}^{\infty} e^{-ct^2/n}\, tdt .$$

Therefore

$$\sum_{|h|>(\alpha n)^\zeta} h\, e^{-ch^2/n} \le 2 \int_{(\alpha n)^\zeta}^{\infty} e^{-ct^2/n}\, tdt$$

and the result follows.

(2.9) LEMMA.

$$\sum_{h=-\infty}^{\infty} |h|^3\, e^{-ch^2/n} \ge 2\left(\int_0^{\infty} e^{-ct^2/n}\, t^3dt - (3n/2ec)^{3/2}\right)$$

$$=(n^2/c^2) - \frac{1}{\sqrt{2}}\,(3n/ec)^{3/2} .$$

For n *sufficiently large,*

$$\sum_{h=-\infty}^{\infty} |h|^3\, e^{-ch^2/n} \ge 2\left(\int_0^{\infty} e^{-ct^2/n}\, t^3dt + (3n/2ec)^{3/2}\right)$$

$$=(n^2/c^2) + \frac{1}{\sqrt{2}}\,(3n/ec)^{3/2} .$$

PROOF.

Let $S = \sum_{h=1}^{\infty} h^3\, e^{-ch^2/n}$ so that

$\sum_{h=-\infty}^{\infty} |h|^3 e^{-ch^2/n} = 2S$. Let $g(t) = t^3 e^{-ct^2/n}$.

Then

$$g'(t) = t^2 e^{-ct^2/n} \left(3 - \frac{2c}{n} t^2\right)$$

and

$$g''(t) = 2t e^{-ct^2/n} \left(2\frac{c}{n} t^2 - 1\right) \left(\frac{c}{n} t^2 - 3\right) .$$

It is easily verified that g is monotone increasing for $0 \le t \le (3n/2c)^{1/2}$, monotone decreasing for $t \ge (3n/2c)^{1/2}$, and takes a maximum value of $(3n/2ec)^{3/2}$ when $t = (3n/2c)^{1/2}$. For $(n/2c)^{1/2} \le t \le (3n/c)^{1/2}$, g is concave downward. Choose an integer h_o such that $h_o - 1 < (3n/2c)^{1/2} \le h_o$. Since

$$h_o - 1 \ge \sqrt{3}\,(n/2c)^{1/2} - 1 \ge (n/2c)^{1/2} \text{ if } n \ge (2+\sqrt{3})c ,$$

and

$$h_o < \frac{1}{\sqrt{2}} (3n/c)^{1/2} + 1 \le (3n/c)^{1/2}$$

if $n \ge \left(2 + \frac{4}{3}\sqrt{2}\right)c$ (which exceeds $(2 + \sqrt{3})c$),

it follows that g is concave downward for $h_o - 1 \le t \le h_o$,

when $n \ge \left(2 + \frac{4}{3}\sqrt{2}\right) c$.

Set $a(t) = h^3 e^{-ch^2/n}$, $h \leq t < h+1$,

$b(t) = (h+1)^3 e^{-c(h+1)^2/n}$, $h \leq t < h+1$.

Now $b(t) < g(t) \leq a(t)$ for $t \geq h_o$,

$a(t) \leq g(t) < b(t)$ for $0 \leq t \leq h_o-1$,

and proceeding just as in the proof of Theorem (2.6) we obtain :

$$\int_0^{h_o} \left(b(t)-a(t)\right)dt = h_o^3 e^{-ch_o^2/n} = g(h_o) \quad .$$

$$\int_0^{\infty} g(t)dt \leq S + g(h_o) + \left\{(3n/2ec)^{3/2} - g(h_o)\right\}$$

$$= S + (3n/2ec)^{3/2}$$

whence

$$\sum_{h=-\infty}^{\infty} |h|^3 e^{-ch^2/n} \geq 2\left(\int_0^{\infty} e^{-ct^2/n} t^3 dt - (3n/2ec)^{1/2}\right)$$

$$= (n^2/c^2) - \frac{1}{\sqrt{2}} (3n/ec)^{3/2} \quad .$$

$$\int_0^{\infty} g(t)dt \geq S - \int_{h_o-1}^{h_o} g(t)dt + 2\left\{\int_{h_o-1}^{h_o} g(t)dt - \frac{g(h_o-1)+g(h_o)}{2}\right\}$$

$$\geq S - (3n/2ec)^{3/2} \quad \text{if} \quad n > \left(2 + \frac{4}{3}\sqrt{2}\right) c$$

(and therefore if $n > 4c$) since g is then concave downward in $h_o-1 \leq t \leq h_o$. Thus for n sufficiently large (certainly if $n > \left(2 + \frac{4}{3}\sqrt{2}\right)c$),

$$\sum_{h=-\infty}^{\infty} |h|^3 e^{-ch^2/n} \le 2\left(\int_0^{\infty} e^{-ct^2/n} t^3 dt + (3n/2ec)^{1/2}\right)$$

$$=(n^2/c^2) + \frac{1}{\sqrt{2}}(3n/ec)^{3/2} .$$

The following corollary is immediate.

(2.10) COROLLARY.

$$\sum_{h=-\infty}^{\infty} |h|^3 e^{-ch^2/n} = O\left(\int_0^{\infty} e^{-ct^2/n} t^3 dt\right) .$$

CHAPTER 3

SUMMABILITY (e,c)

In this chapter a method of summation (e,c) is defined and it is shown that if either $a_n = o(1)$ or $A_n = o(n^{1/2})$ then summability (B,α,β) implies summability $(e,\alpha/2)$. (In the note at the end of the chapter it is observed that these methods are actually equivalent under either condition.)

For $c > 0$, *summability* (e,c) *is defined by saying that* $\sum_{n=o}^{\infty} a_n = A(e,c)$

if $\sqrt{c/\pi n} \sum_{h=-\infty}^{\infty} e^{-ch^2/n} A_{n+h} \to A$ *when* $n \to \infty$.

(Recall that $A_m = 0$ for $m < 0$.)

For this definition and results for Borel summability B analogous to those which follow for summability (B,α,β) see Hardy [8, §9.10] and the references given there. Part (i) of Theorem (3.1) is proved in Borwein [5], while part (ii) is proved (with details omitted) in Hardy [8, Theorem 150].

(3.1) THEOREM.

If $a_n = o(1)$ *and*

either (i) $\sum_{n=o}^{\infty} a_n = A\ (B,\alpha,\beta)$

or (ii) $\sum_{n=o}^{\infty} a_n = A\ (e,c)$,

then $A_n = o(n^{1/2})$.

PROOF.

The result for case (i) is just (1.2) with $\rho = 0$.

Suppose now that condition (ii) holds.

Since

$$\sqrt{c/\pi n} \sum_{h=-\infty}^{\infty} e^{-ch^2/n} = 1 + o(1) \quad \text{by (2.4)} ,$$

it follows that

$$(3.2) \quad \sqrt{c/\pi n} \sum_{h=-\infty}^{\infty} (A_{n+h} - A_n)\, e^{-ch^2/n}$$

$$= A + o(1) - A_n\,\{1 + o(1)\} \quad .$$

Since $a_n \to 0$, it is well known that $A_n = o(n)$.

Thus

$$A_{n+h} = O(n + |h|) \quad \text{by (1.10) with } \lambda = 1 ,$$

and therefore

$$A_{n+h} - A_n = O(n + |h|) + O(n) = O(n + |h|) \quad .$$

Thus

$$\sqrt{c/\pi n} \sum_{|h|>n^\zeta} (A_{n+h} - A_n)\, e^{-ch^2/n}$$

$$= \sqrt{c/\pi n} \sum_{|h|>n^\zeta} O(n + |h|)\, e^{-ch^2/n}$$

$$= O\left\{n^{-1/2} \left(n \sum_{|h|>n^\zeta} e^{-ch^2/n} + \sum_{|h|>n^\zeta} |h|\, e^{-ch^2/n}\right)\right\}$$

$$= O\left\{n^{1/2} \sum_{|h|>n^\zeta} |h|\, e^{-ch^2/n}\right\}$$

$$= O\left\{n^{1/2} \int_{n^\zeta}^{\infty} e^{-ct^2/n}\, t\,dt\right\} \qquad \text{by (2.8) with } \alpha = 1$$

$$= O\left\{n^{1/2} \left[(-n/2c)\, e^{-ct^2/n}\right]_{n^\zeta}^{\infty}\right\}$$

$$= O\left\{n^{3/2}\, e^{-cn^{2\zeta-1}}\right\}$$

$$= o(1) \quad \text{as} \quad n \to \infty \quad \text{since} \quad 2\zeta-1 > 0 \; .$$

Also

$$\sqrt{c/\pi n} \sum_{|h|\leq n^{\zeta}} (A_{n+h} - A_n)\, e^{-ch^2/n}$$

$$= \sqrt{c/\pi}\; n^{-1/2} \sum_{|h|\leq n^{\zeta}} o(|h|)\, e^{-ch^2/n} \qquad \text{by (1.11) with } j = n$$

$$= o\left(n^{-1/2} \sum_{|h|\leq n^{\zeta}} |h|\, e^{-ch^2/n}\right)$$

since (1.11) holds uniformly for $|h| \leq n^{\zeta}$

$$= o\left(n^{-1/2} \int_0^{\infty} e^{-ct^2/n}\, t\,dt\right) \qquad \text{by (2.7)}$$

$$= o\left(n^{-1/2}\, (n/2c)\right) \qquad \text{by (2.1)}$$

$$= o(n^{1/2}) .$$

Therefore

$$\sqrt{c/\pi n} \sum_{h=-\infty}^{\infty} (A_{n+h} - A_n)\, e^{-ch^2/n} = o(1) + o(n^{1/2}) = o(n^{1/2}) .$$

It follows from (3.2) that

$$A_n \left(1 + o(1)\right) = A + o(1) + o(n^{1/2}) = o(n^{1/2}) .$$

Therefore $A_n = o(n^{1/2})$.

We now prove as lemmas four results which will be needed in the proof of Theorem (3.8). In each of these lemmas we let $h = m-n = m-x/\alpha$, choose $\frac{1}{2} < \zeta < \frac{2}{3}$, and assume the condition

(3.3) $A_n = o(n^{1/2})$.

(3.4) LEMMA.

$$e^{-\alpha n} \sum_{|h|>(\alpha n)^\zeta} A_m \frac{(\alpha n)^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)} = o(1) \quad \text{as} \quad n \to \infty .$$

PROOF.

$$e^{-\alpha n} \sum_{|h|>(\alpha n)^\zeta} A_m \frac{(\alpha n)^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)}$$

$$= O\left(e^{-\alpha n} \sum_{|h|>(\alpha n)^\zeta} m^{1/2} \frac{(\alpha n)^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)} \right)$$

since certainly $A_m = O(m^{1/2})$ by (3.3)

$$= O\left((\alpha n)^{1/2} e^{-\alpha n} \sum_{|h|>(\alpha n)^\zeta} \frac{(\alpha n)^{\alpha m+\beta-3/2}}{\Gamma(\alpha m+\beta-1/2)} \right) \quad \text{by (1.7)}$$

$$= O\left((\alpha n)^{1/2} e^{-(\alpha n)^\eta} \right) \quad \text{where } o < \eta < 2\zeta - 1$$

by (1.3) with $x = \alpha n$

$$= o(1) \quad \text{as} \quad n \to \infty.$$

(3.5) LEMMA.

$$\sum_{|h|\le(\alpha n)^\zeta} e^{-\alpha h^2/2n} A_{n+h} \, O\left(\frac{|h|+1}{n}\right) = o(n^{1/2}) \quad \text{as } n \to \infty.$$

PROOF.

$$\sum_{|h|\leq(\alpha n)^{\zeta}} e^{-\alpha h^2/2n} A_{n+h}\ O\left(\frac{|h|+1}{n}\right)$$

$$= \sum_{|h|\leq(\alpha n)^{\zeta}} e^{-\alpha h^2/2n}\ o(n^{1/2})\ O\left(\frac{|h|+1}{n}\right) \quad \text{by (1.12) with } j = n$$

$$= o(n^{-1/2}) \sum_{|h|\leq(\alpha n)^{\zeta}} e^{-\alpha h^2/2n}\ O(|h|+1)$$

since (1.12) holds uniformly for $|h| \leq (\alpha n)^{\zeta}$

$$= o\left(n^{-1/2} \sum_{|h|\leq(\alpha n)^{\zeta}} e^{-\alpha h^2/2n}\ (|h|+1)\right)$$

$$= o\left(n^{-1/2} \int_0^{\infty} e^{-\alpha t^2/2n}\ (t+1)dt\right) \quad \text{by (2.5) and (2.7)}$$

$$= o\left(n^{-1/2}\{(n/\alpha) + (1/2)\sqrt{2\pi n/\alpha}\}\right) \quad \text{by (2.1)}$$

$$= o(n^{1/2}) + o(1)$$

$$= o(n^{1/2}) .$$

(3.6) LEMMA.

$$\sum_{|h|\leq(\alpha n)^{\zeta}} e^{-\alpha h^2/2n} A_{n+h}\ O\left(\frac{|h|^3}{n^2}\right) = o(n^{1/2}) \quad \text{as } n \to \infty.$$

PROOF.

$$\sum_{|h|\le(\alpha n)^{\zeta}} e^{-\alpha h^2/2n} A_{n+h}\, O\left(\frac{|h|^3}{n^2}\right)$$

$$= \sum_{|h|\le(\alpha n)^{\zeta}} e^{-\alpha h^2/2n}\, o(n^{1/2})\, O\left(\frac{|h|^3}{n^2}\right) \qquad \text{by (1.12) with } j = n.$$

$$= o\left\{n^{-3/2} \sum_{|h|\le(\alpha n)^{\zeta}} e^{-\alpha h^2/2n} |h|^3\right\}$$

since (1.12) holds uniformly for $|h| \le (\alpha n)^{\zeta}$

$$= o\left\{n^{-3/2} \int_0^{\infty} e^{-\alpha t^2/2n}\, t^3 dt\right\} \qquad \text{by (2.10)}$$

$$= o\left\{n^{-3/2}(2n^2/\alpha^2)\right\} \qquad \text{by (2.1)}$$

$$= o(n^{1/2}) \quad .$$

(3.7) LEMMA.

$$\sum_{|h|>(\alpha n)^{\zeta}} e^{-\alpha h^2/2n} A_{n+h} = o(1) \qquad \text{as } n \to \infty .$$

PROOF.

$$\sum_{|h|>(\alpha n)^{\zeta}} e^{-\alpha h^2/2n} A_{n+h}$$

$$= \sum_{|h|>(\alpha n)^{\zeta}} e^{-\alpha h^2/2n}\, O(|h|) \qquad \text{by (1.13)}$$

$$= O\left(\sum_{|h|>(\alpha n)^{\zeta}} e^{-\alpha h^2/2n} \; |h| \right)$$

$$= O\left(\int_{(\alpha n)^{\zeta}}^{\infty} e^{-\alpha t^2/2n} \; t dt \right) \qquad \text{by (2.8)}$$

$$= O\left((n/\alpha) \; e^{-\alpha^{2\zeta+1} n^{2\zeta-1}/2} \right) \qquad \text{by (2.2)}$$

$$= O\left(n \; e^{-\rho n^{a}} \right) \qquad \text{where } \rho = \frac{\alpha^{2\zeta+1}}{2} > 0$$

$$\text{and} \quad a = 2\zeta-1 > 0$$

$$= o(1) \text{ as } n \to \infty \; .$$

(3.8) THEOREM.

If $A_n = o(n^{1/2})$, *then* $\sum_{n=o}^{\infty} a_n = A \; (B,\alpha,\beta)$

if and only if $\sum_{n=o}^{\infty} a_n = A \; (e, \; \alpha/2)$.

PROOF.

We first observe that if the theorem holds for $A = 0$ it holds for arbitrary A.

For since

$$\alpha e^{-x} \sum_{m=N}^{\infty} \frac{x^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)} \to 1 \quad \text{as} \quad x \to \infty$$

by the regularity of the (B,α,β) method of summation, we have

$$(3.9)\quad \alpha e^{-x} \sum_{m=N}^{\infty} (A_m - A) \frac{x^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)} \to 0 \quad \text{as} \quad x \to \infty \text{ if and only if}$$

$$(3.10)\quad \alpha e^{-x} \sum_{m=N}^{\infty} A_m \frac{x^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)} \to A \quad \text{as} \quad x \to \infty .$$

Also since

$$\sqrt{\alpha/2\pi n} \sum_{h=-\infty}^{\infty} e^{-\alpha h^2/2n} \to 1 \quad \text{as} \quad n \to \infty \qquad \text{by (2.4)} ,$$

$$(3.11)\quad \sqrt{\alpha/2\pi n} \sum_{h=-\infty}^{\infty} e^{-\alpha h^2/2n} (A_{n+h} - A) \to 0 \quad \text{as} \quad n \to \infty \text{ if and only if}$$

$$(3.12)\quad \sqrt{\alpha/2\pi n} \sum_{h=-\infty}^{\infty} e^{-\alpha h^2/2n} A_{n+h} \to A \quad \text{as} \quad n \to \infty .$$

Thus assuming the result for limit 0, we have (3.10) $\Longleftrightarrow$ (3.9) $\Longleftrightarrow$ (3.11) $\Longleftrightarrow$ (3.12) and the result for limit A follows .

We therefore assume as we may that $A = 0$. We further assume, without loss in generality, that $a_o = \ldots = a_{N-1} = 0$, so that $A_m = 0$ if $m < N$.

Now let $x = \alpha n$, $\frac{1}{2} < \zeta < \frac{2}{3}$, and $h = m-n = m-x/\alpha$.

Then

$$e^{-\alpha n} \sum_{m=N}^{\infty} A_m \frac{(\alpha n)^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)}$$

$$= e^{-\alpha n} \sum_{|h|>(\alpha n)^{\zeta}} A_m \frac{(\alpha n)^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)}$$

$$+ e^{-\alpha n} \sum_{|h|\leq(\alpha n)^{\zeta}} A_m \frac{(\alpha n)^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)}$$

$$= o(1) + n^{-1/2}\left\{\sum_{|h|\leq(\alpha n)^{\zeta}} e^{-\alpha h^2/2n} A_{n+h}\right.$$

$$+ \sum_{|h|\leq(\alpha n)^{\zeta}} e^{-\alpha h^2/2n} A_{n+h}\, O\left(\frac{|h|+1}{n}\right)$$

$$\left.+ \sum_{|h|\leq(\alpha n)^{\zeta}} e^{-\alpha h^2/2n} A_{n+h}\, O\left(\frac{|h|^3}{n^2}\right)\right\}$$

by (3.4) and (1.5) with $x = \alpha n$

$$= o(1) + n^{-1/2} \sum_{|h|\leq(\alpha n)^{\zeta}} e^{-\alpha h^2/2n} A_{n+h} + o(1) + o(1)$$

by (3.5) and (3.6)

$$= n^{-1/2} \sum_{|h|\leq(\alpha n)^{\zeta}} e^{-\alpha h^2/2n} A_{n+h} + o(1) \quad .$$

Since

$$\sum_{|h|>(\alpha n)^{\zeta}} e^{-\alpha h^2/2n} A_{n+h} = o(1) \qquad \text{by (3.7)}$$

it follows that

$$e^{-\alpha n} \sum_{m=N}^{\infty} A_m \frac{(\alpha n)^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)} = o(1) \quad \text{as} \quad n \to \infty \tag{3.13}$$

if and only if

$$n^{-1/2} \sum_{h=-\infty}^{\infty} e^{-\alpha h^2/2n} A_{n+h} = o(1) \quad \text{as } n \to \infty;$$

i.e. if and only if $\sum_{n=o}^{\infty} a_n = 0 \ (e,\alpha/2)$.

Since

$$\sum_{n=o}^{\infty} a_n = 0 \ (B,\alpha,\beta)$$

implies

$$e^{-\alpha n} \sum_{m=N}^{\infty} A_m \frac{(\alpha n)^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)} = o(1) \quad \text{as} \quad n \to \infty$$

the theorem is established in one direction. (The rest of the other direction is established in the note at the end of this chapter.)

(3.14) COROLLARY.

If $a_n = o(1)$, *then* $\sum_{n=o}^{\infty} a_n = A \ (B,\alpha,\beta)$

if and only if $\sum_{n=o}^{\infty} a_n = A \ (e,\ \alpha/2)$

PROOF.

The result follows directly from Theorems (3.1) and (3.8) .

NOTE.

Since it is not needed in the proof of the "O" Tauberian Theorem, a proof of the "if" direction of Theorem (3.8) (*i.e.* that $A_n = o(n^{1/2})$ and summability $(e, \alpha/2)$ imply summability (B,α,β)) has been omitted from the body of the thesis. A proof is included here for the sake of completeness and because of the interest of the result.

After (3.13), it is plainly sufficient to show that, if $e^{-x} \sum_{m=N}^{\infty} A_m \frac{x^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)} \to o$ as $x \to \infty$ through integer multiples of α, then it approaches zero as $x \to \infty$ (without restriction). It is convenient to assume that (1.3) and (1.5) hold with $h = n - [x/\alpha]$ rather than $h = n - x/\alpha$. (Only trivial modifications of Borwein's proof in [5] are needed for this change in h.)

Letting $h = m-n = m - [x/\alpha]$, it follows just as in Lemma (3.4) that

(3.15) $$e^{-x} \sum_{|h|>x^{\zeta}} A_m \frac{x^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)} = o(1) \quad \text{as } x \to \infty .$$

Also, just as in Lemmas (3.5) and (3.6) (with an appropriate modification in Lemma (1.12)), we obtain:

$$(3.16) \quad \sum_{|h|\le x^{\zeta}} e^{-\alpha^2h^2/2x} A_{n+h}\, O\left(\frac{|h|+1}{x}\right) = o(x^{1/2}) \quad \text{as } x \to \infty,$$

and

$$(3.17) \quad \sum_{|h|\le x^{\zeta}} e^{-\alpha^2h^2/2x} A_{n+h}\, O\left(\frac{|h|^3}{x^2}\right) = o(x^{1/2}) \quad \text{as } x \to \infty.$$

Thus, using (3.15), (1.5), (3.16), and (3.17), we obtain:

$$(3.18) \quad e^{-x} \sum_{m=N}^{\infty} A_m \frac{x^{\alpha m+\beta-1}}{\Gamma(\alpha m+\beta)} = \frac{1}{\sqrt{2\pi x}} \sum_{|h|\le x^{\zeta}} A_{n+h}\, e^{-\alpha^2h^2/2x} + o(1)\ .$$

Letting $x = \alpha n + k$ where $o \le k < \alpha$, it follows from (3.18) using Lemmas (3.4), (3.5), (3.6) and (3.7), that it is sufficient to prove that:

$$(\alpha n)^{-1/2} \sum_{|h|\le x^{\zeta}} e^{-\alpha h^2/2n} A_{n+h} = o(1) \quad \text{as} \quad n \to \infty$$

implies

$$x^{-1/2} \sum_{|h|\le x^{\zeta}} e^{-\alpha^2h^2/2x} A_{n+h} = o(1) \quad \text{as} \quad x \to \infty\ .$$

Since $x^{-1/2} = (\alpha n)^{-1/2} + O(n^{-3/2})$

and

$$e^{-\frac{\alpha^2 h^2}{2(\alpha n+k)}} = e^{-\alpha h^2/2n}\left\{1 + 0\left(\frac{h^2}{n^2}\right)\right\} = e^{-\alpha h^2/2n}\left\{1 + 0\left(\frac{|h|+1}{n}\right)\right\},$$

(See the proof of similar results after (4.14))

the result follows.

CHAPTER 4

SUMMABILITY (γ,k)

Our purpose in this chapter is to prove that if

$A_n = o(n^{1/2})$ and $\sum_{n=o}^{\infty} a_n = A\ (e,c)$ then for $0 < d < c$,

$\sum_{n=o}^{\infty} a_n = A(e,d)$ (Theorem 4.23). We do this by first defining

a method of summation (γ,k), establishing some properties of it, and eventually linking summability (γ,k) to summability (e,c) where

$c = \frac{k}{2(1-k)}$ (Theorem 4.18).

For the definition and preliminary discussion of summability (γ,k) see Hardy [8, §9.11]. Results (4.3), (4.4), and (4.24) are stated and proved there, while (4.22) is stated with a proof indicated. Results (4.15), (4.17), (4.18), and (4.23) while not stated explicitly are implicit in the material of section 9.11 cited above.

Let $0 < k < 1$. Suppose $\sum_{n=o}^{\infty} a_n x^n$ is convergent for $|x| < 1$

and let $f(x) = \sum_{n=o}^{\infty} a_n x^n$. The Taylor series expansion for f

about $x_o = 1-k$ as centre, valid at least in a disk of radius k is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(1-k)}{n!} (ky)^n$$

$$= \sum_{n=0}^{\infty} b_n y^n \quad \text{where} \quad x = 1 - k + ky \quad \text{and } b_n = \frac{f^{(n)}(1-k)}{n!} k^n .$$

Summability (γ, k) *is defined by saying that*

$\sum_{n=0}^{\infty} a_n = A\ (\gamma,k)$ *if* $\sum_{n=0}^{\infty} a_n x^n$ *converges for* $|x| < 1$

and $\sum_{n=0}^{\infty} b_n = A$.

Let $B_m = b_0 + b_1 + \ldots + b_m$. Since

$x = 1-k+ky$, $1-k = k(1-y)$ and $\frac{1}{1-y} = \frac{k}{1-x}$.

Thus

$$\sum_{m=0}^{\infty} B_m y^m = \frac{1}{1-y} \sum_{m=0}^{\infty} b_m y^m = \frac{k}{1-x} f(x) = \frac{k}{1-x} \sum_{n=0}^{\infty} a_n x^n$$

$$= k \sum_{n=0}^{\infty} A_n x^n = k \sum_{n=0}^{\infty} A_n (1-k+ky)^n .$$

Now $k\left((1-k) + ky\right)^n = \sum_{r=0}^{n} \binom{n}{r} (1-k)^{n-r} k^{r+1} y^r$.

Here y^m appears only if $n \geq m$ and then the coefficient of y^m is $\binom{n}{m} (1-k)^{n-m} k^{m+1}$.

(4.1) Therefore $B_m = \sum_{n=m}^{\infty} A_n \binom{n}{m} (1-k)^{n-m} k^{m+1}$

$$= k^{m+1} \sum_{n=0}^{\infty} A_{m+n} \binom{m+n}{m} (1-k)^n .$$

Similarly, since

$$\sum_{m=o}^{\infty} b_m y^m = \sum_{n=o}^{\infty} a_n x^n = \sum_{n=o}^{\infty} a_n (1-k+ky)^n ,$$

.2) $$b_m = k^m \sum_{n=m}^{\infty} a_n \binom{n}{m} (1-k)^{n-m}$$

$$= k^m \sum_{n=o}^{\infty} a_{m+n} \binom{m+n}{m} (1-k)^n .$$

.3) THEOREM.

The (γ, k) *method of summation is regular.*

PROOF.

It follows from (4.1) that

$$B_m = \sum_{n=o}^{\infty} c_{m,n} A_n \quad \text{where}$$

$$c_{m,n} = \begin{cases} 0 & , \quad n < m \\ k^{m+1} \binom{n}{m} (1-k)^{n-m} & , \quad n \geq m . \end{cases}$$ Thus

$$\sum_{n=o}^{\infty} |c_{m,n}| = \sum_{n=m}^{\infty} c_{m,n} = k^{m+1} \sum_{n=m}^{\infty} \binom{n}{m} (1-k)^{n-m}$$

$$= k^{m+1} \sum_{n=o}^{\infty} \binom{m+n}{m} (1-k)^n$$

$$= k^{m+1} \left(1 - (1-k)\right)^{-(m+1)} = 1 .$$

$$\lim_{m\to\infty} c_{m,n} = \lim_{m\to\infty} 0 = 0 \quad \text{for all } n.$$

The theorem follows from a well known result (see Hardy [8, Theorem 2]).

Note that (4.1) and (4.2) were obtained using the fact that the radius of convergence R of $\sum_{n=o}^{\infty} a_n x^n$ is 1 (actually ≥ 1 but if $R > 1$, $\sum_{n=o}^{\infty} a_n$ converges and the method is of no interest).

While a method of summability could be defined by requiring that B_m as given by (4.1) exist for all m and tend to A as m tends to ∞, it is convenient for our purpose to keep the restriction and some of the following theorems depend upon it.

(4.4) THEOREM.

If $\sum_{n=o}^{\infty} a_n = A\ (\gamma,k)$ *and* $0 < \ell < k$ *then* $\sum_{n=o}^{\infty} a_n = A\ (\gamma,\ell)$.

PROOF.

Let $x = 1-k + ky = 1-\ell + \ell z$. Then

$ky = k-\ell + \ell z$ and $y = 1 - \frac{\ell}{k} + \frac{\ell}{k} z$ where $0 < \frac{\ell}{k} < 1$.

Write $f(x) = \sum_{n=o}^{\infty} a_n x^n = \sum_{n=o}^{\infty} b_n y^n = \sum_{n=o}^{\infty} c_n z^n$.

If $\sum_{n=o}^{\infty} a_n = A\ (\gamma,k)$ then $\sum_{n=o}^{\infty} b_n = A$

(by the definition of summability (γ,k)) .

Since the $(\gamma,\frac{\ell}{k})$ method is regular $\sum_{n=o}^{\infty} b_n = A\ (\gamma,\frac{\ell}{k})$.

That is $\sum_{n=o}^{\infty} c_n = A$. But this is just summability (γ,ℓ).

Therefore $\sum_{n=o}^{\infty} a_n = A\ (\gamma,\ell)$.

The following theorem gives estimates for summability (γ,k) similar to those given in Chapter 1 for summability (B,α,β). It is essentially Theorem 139 of Hardy [8]. While a proof is outlined there, a much more detailed proof will be given here, because of the interest of the techniques and their similarity to those required to establish the results in Chapter 1, as well as for the sake of completeness.

(4.5) THEOREM.

Suppose that $0 < k < 1$ *and*

$$u_m = u_m(n) = k^{n+1} \binom{m}{n} (1-k)^{m-n} \textit{ for } m \geq n$$

$$(u_m = 0 \quad \text{for} \quad m < n)$$

so that

(4.6) $$\sum_{m=o}^{\infty} u_m = \sum_{m=n}^{\infty} u_m = k^{n+1} \sum_{r=o}^{\infty} \binom{n+r}{n} (1-k)^r$$

$$= k^{n+1} \left(1 - (1-k)\right)^{-(n+1)} = 1 .$$

Then

(1) *the largest* u_m *is* u_M *where* $M = [n/k]$,

two terms u_{M-1} *and* u_M *being equal if* n/k *is an integer;*

(2) *if* $m = M+h$ *and* $0 < \delta < 1$, *then*

$$\sum_{|h|>\delta n} u_m = O\left(e^{-\mu n}\right)$$ *for some positive* $\mu = \mu(k,\delta)$;

(3) *if* $1/2 < \zeta < 2/3$ *then* $\sum_{|h|>n^\zeta} u_m = O\left(e^{-n^\eta}\right)$

where $0 < \eta < 2\zeta-1$;

(4) *if* $|h| \leq n^\zeta$ *and* $c = \frac{k}{2(1-k)}$ *then*

$$(4.7) \quad u_m = \sqrt{c/\pi M}\; e^{-ch^2/M} \left\{1 + O\left(\frac{|h|+1}{n}\right) + O\left(\frac{|h|^3}{n^2}\right)\right\}$$

$$= \sqrt{c/\pi M}\; e^{-ch^2/M} \left\{1 + O\left(\frac{|h|+1}{M}\right) + O\left(\frac{|h|^3}{M^2}\right)\right\};$$

(5) *if* λ *is any real constant*

$$(4.8) \quad \sum_{|h|>n^\zeta} m^\lambda u_m = O\left(e^{-n^\eta}\right) \qquad \text{where } 0 < \eta < 2\zeta-1 .$$

PROOF.

$$(4.9) \quad \frac{u_m}{u_{m-1}} = \frac{\binom{m}{n}(1-k)^{m-n}}{\binom{m-1}{n}(1-k)^{m-n-1}} = \frac{m}{m-n}\quad(1-k) .$$

If $m < \frac{n}{k}$, $km < n$, $m - km > m-n$ and therefore $\frac{m}{m-n}(1-k) > 1$.

Therefore $u_m > u_{m-1}$.

If $m > \frac{n}{k}$, $km > n$, $m-km < m-n$ and thus $\frac{m}{m-n}(1-k) < 1$.

Therefore $u_m < u_{m-1}$.

If $m = \frac{n}{k}$ $(=M)$ then $\frac{m}{m-n}(1-k) = 1$ and $u_M = u_{M-1}$.

This proves part (1).

Now choose n sufficiently large that $n^\zeta < \delta n$ and $n-M < -\delta n$. (We may suppose $0 < \delta < \frac{1}{k} - 1$ since decreasing δ increases the strength of (2). Then if n is sufficiently large $\delta < \frac{1}{k} - 1 - \frac{1}{n}$.

$$\text{Therefore} \quad n+1 < \frac{n}{k} - \delta n ,$$

$$n - (\frac{n}{k} - 1) < -\delta n ,$$

$$\text{and} \quad n-M < -\delta n .$$

Then S_1 in what follows does not disappear.)

$$\sum_{h=-\infty}^{\infty} u_{M+h}$$

$$= \sum_{-M\leq h<-\delta n} u_{M+h} + \sum_{-\delta n\leq h<-n^\zeta} u_{M+h} + \sum_{|h|\leq n^\zeta} u_{M+h}$$

$$+ \sum_{n^\zeta<h\leq\delta n} u_{M+h} + \sum_{h>\delta n} u_{M+h}$$

$$= \sum_{m=o}^{M_1-1} u_m + \sum_{m=M_1}^{M_2-1} u_m + \sum_{m=M_2}^{M_3} u_m$$

$$+ \sum_{m=M_3+1}^{M_4} u_m + \sum_{m=M_4+1}^{\infty} u_m$$

$$= S_1 + S_2 + S_3 + S_4 + S_5$$

where $M_1 = M - [\delta n]$, $M_2 = M - [n^\zeta]$, $M_3 = M + [n^\zeta]$,

and $M_4 = M + [\delta n]$.

Then

$$S_1' = \sum_{m=o}^{M_1-1} m^\lambda u_m \leq \sum_{m=o}^{M_1-1} M_1^{\ \lambda} u_{M_1} \leq M_1^{\ \lambda+1} u_{M_1} \leq M^{\lambda+1} u_{M_1} .$$

Therefore $S_1' = 0\left(n^{\lambda+1} u_{M_1}\right)$.

$$S_2' = \sum_{m=M_1}^{M_2-1} m^\lambda u_m \leq \sum_{m=M_1}^{M_2} M_2^{\ \lambda} u_{M_2} \leq M_2^{\ \lambda+1} u_{M_2} \leq M^{\lambda+1} u_{M_2} .$$

Therefore $S_2' = 0\left(n^{\lambda+1} u_{M_2}\right)$.

$$S_4' = \sum_{m=M_3+1}^{M_4} m^\lambda u_m \leq \sum_{m=M_3+1}^{M_4} M_4^{\ \lambda} u_{M_3} \leq M_4^{\ \lambda+1} u_{M_3} .$$

Therefore $S_4' = 0\left(n^{\lambda+1} u_{M_3}\right)$, since $M_4 = 0(n)$.

$$S_5{}' = \sum_{m=M_4+1}^{\infty} m^\lambda u_m = (H+1)^\lambda u_{H+1} + (H+2)^\lambda u_{H+2} + \dots$$

where $H = M_4$.

Since $H+2 > \frac{n}{k} + \delta n$, if $m > H+1$ then $m > \frac{n}{k}(1+\delta k)$.

Therefore $-n(1+\delta k) > -mk$

and $(1+\delta k)m - (1+\delta k)n > (1+\delta k)m - mk$;

i.e. $(1+\delta k)(m-n) > m(1-k+\delta k)$.

Therefore $\frac{m}{m-n} < \frac{1+\delta k}{1-k+\delta k}$.

Thus by (4.9)

$$\frac{u_m}{u_{m-1}} = \frac{m}{m-n}(1-k) < \frac{(1+\delta k)(1-k)}{1-k+\delta k}$$

$$= 1 - \frac{k^2\delta}{1-k+\delta k} = 1-\delta_1 = r_1 .$$

Thus if $m > H+1$, $\frac{u_m}{u_{m-1}} < r_1$ where

$1-k < r_1 < 1$ since $\delta_1 < \frac{k^2\delta}{k\delta} = k$ and $\delta_1 > 0$.

Also since $(H+r)^\lambda = r^\delta \left(1+\frac{H}{r}\right)^\lambda \leq r^\lambda 2^\lambda$ if $r \geq H$,

while $(H+r)^\lambda \leq (2H)^\lambda$ if $r \leq H$,

we have $(H+r)^\lambda \leq (2H)^\lambda r^\lambda$.

Therefore

$$S_5' \leq (2H)^\lambda \, u_{H+1} \{1^\lambda + 2^\lambda r_1 + 3^\lambda r_1{}^2 + 4^\lambda r_1{}^3 + \dots\}$$

$$\leq (2H)^\lambda \, u_H \left(\sum_{n=1}^{\infty} n^\lambda r_1{}^{n-1}\right)$$

$$= O\left(H^\lambda \, u_H\right) \qquad \text{since the series converges}$$

$$= O\left(M_4{}^\lambda \, u_{M_4}\right)$$

$$= O\left(n^\lambda \, u_{M_4}\right) \quad .$$

Taking $\lambda = 0$ in the S_i' , $i = 1,2,4,5$, gives

$$S_1 = O\left(n \, u_{M_1}\right) , \quad S_2 = O\left(n \, u_{M_2}\right) , \quad S_4 = O\left(n \, u_{M_3}\right) , \quad S_5 = O\left(u_{M_4}\right) .$$

Since $n^\lambda e^{-\mu n} = O\left(e^{-\mu_1 n}\right)$ if $0 < \mu_1 < \mu$,

it therefore suffices to prove that u_{M_1} and u_{M_4} are $O(e^{-\mu n})$ to establish part (2) .

Since $e^{-\mu n} = O\left(e^{-n^\eta}\right)$ it suffices to prove that

that u_{M_2} and u_{M_3} are $O\left(e^{-n^\eta}\right)$ to establish parts (3) and (5) .

Now $\ell^{n+1} \sum_{m=n}^{\infty} \binom{m}{n} (1-\ell)^{m-n} = 1$ for all ℓ

with $0 < \ell < 1$.

Therefore $0 < \ell^{n+1} \binom{m}{n} (1-\ell)^{m-n} < 1$.

Thus

$$u_m = k^{n+1} \binom{m}{n} (1-k)^{m-n} < \frac{k^{n+1} \binom{m}{n} (1-k)^{m-n}}{\ell^{n+1} \binom{m}{n} (1-\ell)^{m-n}} = \left(\frac{k}{\ell}\right)^{n+1} \left(\frac{1-k}{1-\ell}\right)^{m-n} .$$

For either $m = M_1$ or M_4 ; *i.e.* for $m = M \mp [\delta n]$,

$$\left(\frac{k}{\ell}\right)^{n+1} \left(\frac{1-k}{1-\ell}\right)^{m-n}$$

$$= \frac{k}{\ell} \left(\frac{1-k}{1-\ell}\right)^{M-(n/k)\pm\{\delta n-[\delta n]\}} \left\{\left(\frac{k}{\ell}\right)^n \left(\frac{1-k}{1-\ell}\right)^{n/k)-n\mp\delta n}\right\}$$

$$= \frac{k}{\ell}\left(\frac{1-k}{1-\ell}\right)^a \Theta^n \qquad \text{where } -2 < a < 1$$

$$= O(\Theta^n) \qquad \text{where } \Theta = \Theta(\ell) = \frac{k}{\ell}\left(\frac{1-k}{1-\ell}\right)^{(1/k)-1\mp\delta} .$$

$\Theta(\ell) = 1$ when $\ell = k$. $\Theta(\ell) \to +\infty$ as $\ell \downarrow 0$ and as $\ell \uparrow 1$.

$$\Theta'(\ell) = (1-k)^{(1/k)-1\mp\delta} \frac{\ell(1 \mp \delta k) - k}{\ell^2(1-\ell)^{(1/k)\mp\delta}} .$$

Therefore $\Theta'(\ell) = 0$ when $\ell = \dfrac{k}{1 \mp \delta k}$.

Thus $\Theta(\ell)$ has a single minimum when $\ell = \dfrac{k}{1 \mp \delta k}$.

$\Theta'(k) = \frac{\mp\delta}{1-k} \neq 0$. Hence we can choose ℓ such that $\Theta(\ell) < 1$. (For $m = M_1$, $\Theta'(k) = \frac{-\delta}{1-k} < 0$ so that we can choose $\ell > k$ such that $\Theta(\ell) < 1$. For $m = M_4$ $\Theta'(k) = \frac{\delta}{1-k} > 0$ so we can choose $\ell < k$ such that $\Theta(\ell) < 1$.)

Therefore u_{M_1} and u_{M_4} are $O(\Theta_o{}^n)$ where $0 < \Theta_o < 1$.

Let $\mu = \log \frac{1}{\Theta_o}$. Then $\mu > 0$ and $\Theta_o{}^n = e^{n \log \Theta_o} = e^{-\mu n}$.

Thus u_{M_1} and u_{M_4} are $O(e^{-\mu n})$ where $\mu > 0$ and part (2) is established.

We next prove part (4). Suppose $|h| \leq n^{\zeta}$ and write $M = [n/k] = n/k - f$ where $0 \leq f < 1$.

Then $m = M+h = n/k - f + h$.

(10) Now

$$\log u_m = (n+1) \log k + (m-n) \log (1-k) + \log \Gamma(m+1)$$
$$- \log \Gamma(n+1) - \log \Gamma(m-n+1) \quad .$$

(11) $\log \Gamma(n+1) = (1/2)\log 2\pi - n + (n+1/2) \log n + O(1/n)$ by (1.6) .

$$\log \Gamma(m+1) = (1/2)\log 2\pi - (n/k) + f - h$$
$$+ (m+1/2) \log(n/k) \left[1 + \frac{k(h-f)}{n}\right] + O(\tfrac{1}{m}) \quad .$$

Since $\frac{1}{m} = \frac{1}{n} \frac{k}{1+\frac{k}{n}(h-f)}$ and $1 + \frac{k}{n}(h-f) \to 1$ as $n \to \infty$

for all h such that $|h| \leq n^{\zeta}$, it follows that $O(\frac{1}{m}) = O(\frac{1}{n})$.

Therefore $\log \Gamma(m+1)$

$$= (1/2)\log 2\pi - (n/k) + f - h + (m+1/2)\log n - (m+1/2)\log k$$

$$+ ((n/k) - f + h + 1/2)\log \left[1 + \frac{k(h-f)}{n}\right] + O(\frac{1}{n})$$

$$= (1/2)\log 2\pi - (n/k) + f - h + (m+1/2)(\log n - \log k)$$

$$+ ((n/k) - f + h + 1/2)\left\{\frac{k}{n}(h-f) - \frac{k^2}{2n^2}(h-f)^2 + O\left[\frac{h^3}{n^3}\right]\right\}$$

$$+ O(\frac{1}{n}) \qquad \text{by } (1.9)$$

$$= (1/2)\log 2\pi - (n/k) + (m+1/2)(\log n - \log k)$$

$$+ \frac{k}{2n}(h-f) + \frac{k}{2n}(h-f)^2 - \frac{k^2}{4n^2}(h-f)^2\{1 + 2(h-f)\}$$

$$+ O\left[\frac{|h|^3}{n^2}\right] + O\left[\frac{|h|^3}{n^3}\right] + O\left[\frac{|h|^4}{n^3}\right] + O(\frac{1}{n}) .$$

Now $O\left[\frac{|h|^3}{n^3}\right] = O\left[\frac{|h|+1}{n}\right]$ and $O\left[\frac{|h|^4}{n^3}\right] = O\left[\frac{|h|^3}{n^2}\right]$.

Also $\frac{k}{2n}(h-f) = O\left[\frac{|h|+1}{n}\right]$, $\frac{k}{2n}(h-f)^2 = \frac{kh^2}{2n} + O\left[\frac{|h|+1}{n}\right]$,

and

$$\frac{k^2}{4n^2}(h-f)^2\,(1 + 2(h-f)) = O\left[\frac{|h|+1}{n}\right] + O\left[\frac{|h|^3}{n^2}\right]$$

Thus

(4.12) $$\log \Gamma(m+1) = (1/2)\log 2\pi - \frac{n}{k} + (m+1/2)(\log n - \log k)$$

$$+ \frac{kh^2}{2n} + 0\left(\frac{|h|+1}{n}\right) + 0\left(\frac{|h|^3}{n^2}\right) .$$

Similarly,

$$\log \Gamma(m-n+1) = (1/2)\log 2\pi - \frac{n}{k} + n + f - h$$

$$+ (m-n+1/2)\log\left\{\left(\frac{n(1-k)}{k}\right)\left(1 + \frac{k}{1-k}\frac{h-f}{n}\right)\right\} + 0\left(\frac{1}{m-n}\right) .$$

Since $\frac{1}{m-n} = \frac{1}{n}\left(\frac{1}{\frac{1-k}{k} + \frac{h-f}{n}}\right)$ and $\frac{1}{\frac{1-k}{k} + \frac{h-f}{n}} \to \frac{k}{1-k}$

as $n \to \infty$ for all h such that $|h| \leq n^\zeta$,

$0(\frac{1}{m-n}) = 0(\frac{1}{n})$.

Therefore

$$\log \Gamma(m-n+1) = (1/2)\log 2\pi - \frac{n}{k} + n + f - h$$

$$+ (m-n+1/2)\{\log n - \log k + \log(1-k)\}$$

$$+ \left\{\frac{n(1-k)}{k} + h - f + 1/2\right\}\left(\frac{k}{1-k}\frac{h-f}{n} - \frac{k^2}{2(1-k)^2}\frac{(h-f)^2}{n^2} + 0\left(\frac{|h|^3}{n^3}\right)\right)$$

$$+ 0\left(\frac{1}{n}\right) \quad \text{by } (1.9) .$$

Thus

(4.13) $$\log \Gamma(m-n+1) = (1/2)\log 2\pi - \frac{n}{k} + n$$

$$+ (m-n+1/2)(\log n - \log k + \log(1-k))$$

$$+ \frac{kh^2}{2(1-k)n} + 0\left(\frac{|h|+1}{n}\right) + 0\left(\frac{|h|^3}{n^2}\right) .$$

Combining (4.10), (4.11), (4.12) and (4.13), we obtain

$$\log u_m = -(1/2)\log 2\pi - (1/2)\log n + \log k - (1/2)\log(1-k)$$

$$- \frac{k^2h^2}{2(1-k)n} + O\left(\frac{|h|+1}{n}\right) + O\left(\frac{|h|^3}{n^2}\right) .$$

Thus

$$u_m = \frac{k}{\sqrt{2\pi n(1-k)}} \; e^{-k^2h^2/2(1-k)n} \left\{1 + O\left(\frac{|h|+1}{n}\right) + O\left(\frac{|h|^3}{n^2}\right)\right\}$$

by (1.8)

(4.14) $$= \sqrt{\frac{c}{\pi(n/k)}} \; e^{-\frac{ch^2}{n/k}} \left\{1 + O\left(\frac{|h|+1}{n}\right) + O\left(\frac{|h|^3}{n^2}\right)\right\} .$$

Now $\left|\frac{1}{\sqrt{n/k}} - \frac{1}{\sqrt{M}}\right| = \left|\frac{\sqrt{M} - \sqrt{n/k}}{\sqrt{n/k}\,\sqrt{M}}\right| = \left|\frac{\sqrt{M} - \sqrt{n/k}}{\sqrt{n/k}\,\sqrt{M}} \left(\frac{\sqrt{M} + \sqrt{n/k}}{\sqrt{M} + \sqrt{n/k}}\right)\right|$

$$= \frac{f}{\sqrt{n/k}\,M + (n/k)\sqrt{M}} \le \frac{f}{2M^{3/2}} = O\left(n^{-3/2}\right) .$$

Thus

$$(n/k)^{-1/2} = M^{-1/2} + O\left(n^{-3/2}\right) .$$

Also since $\left|\frac{ch^2}{M} - \frac{ch^2}{n/k}\right| = \frac{ch^2f}{M(n/k)} = O(h^2/n^2)$

$$e^{-\frac{ch^2}{n/k}} = e^{-ch^2/M} \; e^{\left(\frac{ch^2}{M} - \frac{ch^2}{n/k}\right)} = e^{-ch^2/M} \; e^{O(h^2/n^2)}$$

$$= e^{-ch^2/M} \; \{1 + O(h^2/n^2)\} \text{ by } (1.8) .$$

Therefore

$$\sqrt{\frac{c}{\pi n/k}} \quad e^{-\frac{ch^2}{n/k}}$$

$$= \sqrt{c/\pi M} \quad e^{-ch^2/M} \left\{1 + 0\left(\frac{1}{n^{3/2}}\right) + 0(h^2/n^2) + 0\left(\frac{h^2}{n^{7/2}}\right)\right\}$$

$$= \sqrt{c/\pi M} \quad e^{-ch^2/M} \left\{1 + 0\left(\frac{|h|+1}{n}\right)\right\} \quad .$$

Thus we obtain:

(4.7) $$u_m = \sqrt{c/\pi M} \quad e^{-ch^2/M} \left\{1 + 0\left(\frac{|h|+1}{n}\right) + 0\left(\frac{|h|^3}{n^2}\right)\right\}$$

$$= \sqrt{c/\pi M} \quad e^{-ch^2M} \left\{1 + 0\left(\frac{|h|+1}{M}\right) + 0\left(\frac{|h|^3}{M^2}\right)\right\}$$

$$\text{since} \quad 0(\tfrac{1}{n}) = 0(\tfrac{1}{M}) \quad .$$

Thus part (4) is established.

Now $M_2 = M - [n^\zeta]$ and $M_3 + [n^\zeta]$. Since both $\frac{|h|+1}{n}$ and $\frac{|h|^3}{n^2}$ are $0(1)$ for $|h| \leq n^\zeta$, it follows from (4.14) that both u_{M_2} and u_{M_3} are $0\left(n^{-1/2} \; e^{-ck[n^\zeta]^2/n}\right)$.

Let $n^\zeta = [n^\zeta] + q$ where $0 \leq q < 1$.

Then $[n^\zeta]^2 = n^{2\zeta} - 2qn^\zeta + q^2$,

and

$$e^{-ck[n^\zeta]^2/n} = e^{-ckn^{2\zeta-1}} \; e^{-ck(-2qn^\zeta+q^2)/n}$$

$$= 0\left(e^{-ckn^{2\zeta-1}}\right)$$

since $e^{-ck(-2qn^\zeta+q^2)/n} \to 1$ as $n \to \infty$.

Thus u_{M_2} and u_{M_3} are each $0\left(n^{-1/2}\, e^{-ckn^{2\zeta-1}}\right)$ and

thus $0\left(e^{-n^\eta}\right)$ for $0 < \eta < 2\zeta-1$.

Thus parts (3) and (5) of the theorem are established and the theorem is proved.

(4.15) THEOREM.

If $a_n = o(1)$ *and* $\sum_{n=o}^{\infty} a_n = A\ (\gamma,k)$ *then*

$A_n = o(n^{1/2})$.

PROOF.

We use the same notation as earlier in this chapter.

Since $\sum_{n=o}^{\infty} a_n = A\ (\gamma,k)$, $B_n = A + o(1)$ as $n \to \infty$.

$$B_n = k^{n+1} \sum_{m=n}^{\infty} \binom{m}{n}(1-k)^{m-n} A_m = \sum_{m=n}^{\infty} u_m(n)\ A_m$$

$$= \sum_{h=n-M}^{\infty} u_{M+h}\ A_{M+h} = \sum_{h=-\infty}^{\infty} u_{M+h}\ A_{M+h} \quad .$$

Since $\sum_{h=-\infty}^{\infty} u_{M+h} = \sum_{m=n}^{\infty} u_m(n) = 1$ by (4.6)

$$\sum_{h=-\infty}^{\infty} u_{M+h}\,(A_{M+h} - A_M) = B_n - A_M = A + o(1) - A_M \quad . \tag{4.16}$$

Since $a_n = o(1)$ it follows that $A_n = o(n)$ and thus $A_n = O(n)$.

Therefore

$$\sum_{|h|>n^\zeta} u_{M+h}\,(A_{M+h} - A_M)$$

$$= O\left(\sum_{|h|>n^\zeta} (M+h)\, u_{M+h}\right) + O\left(M \sum_{|h|>n^\zeta} u_{M+h}\right)$$

$$= O\left(e^{-n^\eta}\right) + O\left(n\, e^{-n^\eta}\right), \quad \text{by (4.8) with } \lambda = 1 \text{ and } \lambda = 0 \text{ ,}$$

$$= o(1) \quad \text{as} \quad n \to \infty \qquad (\text{or } M \to \infty) \text{ .}$$

Also

$$\sum_{|h|\le n^\zeta} u_{M+h}\,(A_{M+h} - A_M)$$

$$= \sum_{|h|\le n^\zeta} u_{M+h}\, o(|h|) \quad \text{by (1.11) with } j = M$$

$$= o\left(\sum_{|h|\le n^\zeta} u_{M+h}\, |h|\right) \quad \text{since (1.11) holds uniformly}$$

$$\text{for } |h| \le n^\zeta$$

$$= o\left(\sum_{|h|\le n^\zeta} \sqrt{c/\pi M}\;|h|\;e^{-ch^2/M}\right)$$ by (4.7), since for

$|h| \le n^\zeta$, $\frac{|h|+1}{n}$ and $\frac{|h|^3}{n^2}$ are each $O(1)$

$$= o\left(M^{-1/2} \sum_{|h|\le n^\zeta} |h|\;e^{-ch^2/M}\right)$$

$$= o\left(M^{-1/2} \int_0^\infty e^{-ct^2/M}\,t\,dt\right)$$ by (2.7)

$$= o\left(M^{-1/2}\,(M/c) \int_0^\infty e^{-u^2}\,u\,du\right)$$

$$= o(m^{1/2}) \text{ as } M \to \infty.$$

It follows from (4.16) that

$$A + o(1) - A_M = o(1) + o(M^{1/2}) = o(M^{1/2}) .$$

Therefore $A_M = o(M^{1/2})$ and thus $A_M = o(n^{1/2})$.

Since $a_n = o(1)$ and there are at most $[(1/k)+1]$ terms between $[n/k]$ and $[(n+1)/k]$, it follows that $A_n = o(n^{1/2})$.

(4.17) LEMMA.

If $A_n = o(n^{1/2})$ *and* R *is the radius of convergence of* $\sum_{n=o}^{\infty} a_n x^n$ *then* $R \ge 1$.

PROOF.

Since $A_n = o(n^{1/2})$,

$$|a_n| = |A_n - A_{n-1}| \leq |A_n| + |A_{n-1}| = o(n^{1/2}) + o(n^{1/2}).$$

Therefore $|a_n| = o(n^{1/2})$.

Thus $|a_n| \leq n^{1/2}$ for $n \geq N_o$ and thus

$$|a_n|^{1/n} \leq n^{1/2n} = \sqrt{n^{1/n}} .$$

$$\text{Therefore } \limsup_{n \to \infty} |a_n|^{1/n} \leq \limsup_{n \to \infty} \sqrt{n^{1/n}}$$

$$= \lim_{n\to\infty} \sqrt{n^{1/n}} = 1 .$$

$$\text{Thus } R = \frac{1}{\limsup_{n \to \infty} |a_n|^{1/n}} \geq 1 .$$

Thus the condition $R \geq 1$ required by our definition of summability (γ,k) holds throughout the following theorem and need not be checked in the proof.

(4.18) THEOREM.

If $A_n = o(n^{1/2})$ *then* $\sum_{n=o}^{\infty} a_n = A\ (\gamma,k)$ *if and only if* $\sum_{n=o}^{\infty} a_n = A\ (e,c)$ *where* $c = \frac{k}{2(1-k)}$.

PROOF.

That we may assume $A = 0$ without loss of generality here follows from the regularity of the methods in exactly the same way that it did in Theorem (3.8). We therefore take $A = 0$. In the notation we have used throughout this chapter, we must show that

$$\sum_{h=-\infty}^{\infty} u_{M+h} A_{M+h} = o(1) \quad \text{as} \quad n \to \infty \text{ if and only if}$$

$$\sqrt{c/\pi n} \sum_{h=-\infty}^{\infty} e^{-ch^2/n} A_{n+h} = o(1) \quad \text{as } n \to \infty .$$

Now

$$\sum_{|h|>n^\zeta} u_{M+h} A_{M+h} = \sum_{|h|>n^\zeta} u_{M+h} O(M+h) \quad \text{since } A_n = O(n)$$

$$= O\left(\sum_{|h|>n^\zeta} (M+h) u_{M+h} \right)$$

$$= O\left(e^{-n^\eta} \right) \qquad \text{by (4.8) with } \lambda = 1$$

$$= o(1) \quad \text{as} \quad n \to \infty .$$

Therefore

(4.19) $$\sum_{|h|>n^\zeta} u_{M+h} A_{M+h} = o(1) \text{ as } n \to \infty .$$

Also

$$\sum_{|h|\leq n^\zeta} u_{M+h}\, A_{M+h}$$

$$= \sum_{|h|\leq n^\zeta} \sqrt{c/\pi M}\, e^{-ch^2/M} A_{M+h} \left\{1 + O\left(\frac{|h|+1}{M}\right) + O\left(\frac{|h|^3}{M^2}\right)\right\}$$

by (4.7)

$$= \sqrt{c/\pi M} \sum_{|h|\leq n^\zeta} e^{-ch^2/M} A_{M+h} + S + T$$

where

$$S = \sqrt{c/\pi M} \sum_{|h|\leq n^\zeta} e^{-ch^2/M} A_{M+h}\, O\left(\frac{|h|+1}{M}\right)$$

$$= \sqrt{c/\pi M} \sum_{|h|\leq n^\zeta} e^{-ch^2/M}\, o(M^{1/2})\, O\left(\frac{|h|+1}{M}\right)$$

by (1.12) with $j = M$

$$= o\left\{M^{-1} \sum_{|h|\leq n^\zeta} e^{-ch^2/M} (|h|+1)\right\}$$

since (1.12) holds uniformly for $|h| \leq n^\zeta$

$$= o\left[M^{-1} \int_0^\infty e^{-ct^2/M} (t+1)dt\right] \qquad \text{by (2.5) and (2.7)}$$

$$= o\left[M^{-1}(\sqrt{M/2c} + \sqrt{M/c}\cdot\sqrt{\pi}/2)\right]$$

$$= o(1) + o(M^{-1/2})$$

$$= o(1) \text{ as } M \to \infty \quad ,$$

and

$$T = \sqrt{c/\pi M} \sum_{|h|\leq n^\zeta} e^{-ch^2/M} A_{M+h}\, O\left(\frac{|h|^3}{M^2}\right)$$

$$= o\left\{M^{-2} \sum_{|h| \le n^\zeta} e^{-ch^2/M} |h|^3\right\} \qquad \text{as for S above}$$

$$= o\left\{M^{-2} \int_0^\infty e^{-ct^2/M} t^3 dt\right\} \qquad \text{by (2.10)}$$

$$= o\left\{M^{-2} (M/c)^2 (1/2)\right\}$$

$$= o(1) \quad \text{as } M \to \infty.$$

Thus

$$\sum_{|h| \le n^\zeta} u_{M+h} A_{M+h} = \sqrt{c/\pi M} \sum_{|h| \le n^\zeta} e^{-ch^2/M} A_{M+h} + o(1) \quad . \tag{4.20}$$

Now

$$\sqrt{c/\pi M} \sum_{|h| > n^\zeta} e^{-ch^2/M} A_{M+h}$$

$$= \sqrt{c/\pi M} \sum_{|h| > n^\zeta} e^{-ch^2/M} O(|h|)$$

by (1.13) with $\alpha = 1$ and $j = M$

$$= O\left\{M^{-1/2} \sum_{|h| > n^\zeta} e^{-ch^2/M} |h|\right\}$$

$$= O\left\{M^{-1/2} \int_{n^\zeta}^\infty e^{-ct^2/M} t dt\right\} \qquad \text{by (2.8) with } \alpha = 1$$

$$= O\left\{M^{-1/2} \left[-(M/2c) e^{-ct^2/M}\right]_{n^\zeta}^{\infty}\right\}$$

$$= O\left\{M^{-1/2} \left\{(M/2c) e^{-cn^{2\zeta}/M}\right\}\right\}$$

$$= O\left(M^{1/2}\, e^{-\frac{cn^2}{n/k-f}}\right) \qquad \text{where} \quad M = [n/k] = n/k - f,\ 0 \le f < 1$$

$$= O\left(M^{1/2}\, e^{-ckn^{2\zeta-1}}\right)$$

since $n/k - f < n/k$ and therefore $\frac{-1}{n/k-f} < \frac{-1}{n/k}$

$$= O\left(n^{1/2}\, e^{-\rho n^{a}}\right) \qquad \text{where} \quad \rho = ck > 0 \quad \text{and} \quad a = 2\zeta-1 > 0$$

$$= o(1) \quad \text{as} \quad M \to \infty.$$

Therefore

$$\sqrt{c/\pi M} \sum_{|h|>n^\zeta} e^{-ch^2/M} A_{M+h} = o(1) \quad \text{as} \quad M \to \infty. \tag{4.21}$$

Thus

$$\sum_{h=-\infty}^{\infty} u_{M+h} A_{M+h} = o(1) \quad \text{if and only if}$$

$$\sum_{|h|\le n^\zeta} u_{M+h} A_{M+h} = o(1) \qquad \text{(by (4.19)), if and only if}$$

$$\sqrt{c/\pi M} \sum_{|h|\le n^\zeta} e^{-ch^2/M} A_{M+h} = o(1) \quad \text{(by (4.20)), if and only if}$$

$$\sqrt{c/\pi M} \sum_{h=-\infty}^{\infty} e^{-ch^2/M} A_{M+h} = o(1) \qquad \text{(by (4.21)), if and only if}$$

$$\sqrt{c/\pi n} \sum_{h=-\infty}^{\infty} e^{-ch^2/n} A_{n+h} = o(1) \qquad \text{(replacing M by n)} \quad .$$

Thus

$$\sum_{n=o}^{\infty} a_n = A\ (\gamma,k) \text{ if and only if } \sum_{n=o}^{\infty} a_n = A \left[e,\ \frac{k}{2(1-k)}\right]$$

and the theorem is proved.

(4.22) COROLLARY.

If $a_n = o(1)$ *then* $\sum_{n=o}^{\infty} a_n = A\ (\gamma,k)$ *if and only if* $\sum_{n=o}^{\infty} a_n = A\ (e,c)$ *where* $c = \dfrac{k}{2(1-k)}$.

When k increases from 0 to 1, c increases from 0 to ∞. Combining Theorems (4.4) and (4.18) we thus obtain:

(4.23) THEOREM.

If $A_n = o(n^{1/2})$ *and* $\sum_{n=o}^{\infty} a_n = A\ (e,c)$ *then for* $0 < d < c$, $\sum_{n=o}^{\infty} a_n = A\ (e,d)$.

(4.24) COROLLARY.

If $a_n = o(1)$ *and* $\sum_{n=o}^{\infty} a_n = A\ (e,c)$ *then for* $0 < d < c$, $\sum_{n=o}^{\infty} a_n = A\ (e,d)$.

PROOF.

Just combine Theorems (4.23) and (3.1) .

CHAPTER 5

VITALI'S THEOREM

In this chapter we state and prove a theorem due to Vitali which is of critical importance in the proof of the Tauberian Theorem in the next chapter (see Littlewood [11, p. 117] or Titchmarsh [13, p. 168]).

Let D be a region (*i.e.* an open connected subset) of the complex plane. The functions f of a family of functions F are said to be *almost uniformly bounded* in D if for each compact subset A contained in D, there exists an M such that $|f(z)| \leq M$ for all $z \in A$ and for all $f \in F$. Similarly a sequence $\{f_n\}$ of functions $f_n(z)$ is said to *converge almost uniformly* to f(z) in D (and we write $f_n(z) \rightrightarrows f(z)$) if $\{f_n\}$ converges uniformly to f on every compact subset of D. We shall call a family F of functions in D a *normal family* if every sequence $\{f_n\}$ of functions $f_n \in F$ contains an almost uniformly convergent subsequence. For these definitions see Saks and Zygmund [12] and Ahlfors [1].

(5.1) THEOREM. (Vitali)

Let D *be a region of the complex plane and suppose:*

(i) $\{\phi_n\}$ *is a sequence of functions analytic and almost uniformly bounded in* D,

and

(ii) *there exists a sequence* $\{z_m\}$ *(of distinct* z's*)* *in* D *with at least one limit point* $z_o \in D$ *such that* $\lim\limits_{n\to\infty} \phi_n(z_m)$ *exists (necessarily finite because of* (i) *)* *for* $m = 1,2,3,\ldots$.

Then there exists a function $\phi(z)$, *analytic in* D, *such that* $\phi_n(z) \rightrightarrows \phi(z)$.

PROOF.

Condition (i) implies that the sequence $\{\phi_n\}$ is a normal family. (This is the theorem of Stieltjes - Osgood. See Saks and Zygmund [12, p. 119] or Ahlfors [1, Theorem 12 on p. 216].) We first claim that $\{\phi_n\}$ converges pointwise for all z in D. If not there exists a $z^* \in D$ for which $\{\phi_n(z^*)\}$ does not tend to a limit. Then we can find two subsequences of the positive integers $\{n_k'\}$ and $\{n_k''\}$ through which $\{\phi_n(z^*)\}$ tends respectively to two values differing by $c \neq 0$. (Any bounded sequence has a convergent subsequence. If every convergent subsequence converged to the same value $\{\phi_n(z^*)\}$ would converge.) Let

$f_k(z) = \phi_{n_k'}(z) - \phi_{n_k''}(z)$. Then $\{f_k\}$ is a sequence of analytic and almost uniformly bounded functions in D. Thus (again by the theorem of Stieltjes - Osgood) $\{f_k(z)\}$ is a normal family and hence has an almost uniformly convergent subsequence. Let $f(z)$ be the limit of this almost uniformly convergent subsequence. Then $f(z)$ is analytic by a theorem of Weierstrass. (See Saks and Zygmund [12, p. 116] or Ahlfors [1, Theorem 1 on p. 174].) Since $\{\phi_n(z_m)\}$ converges as $n \to \infty$ for $m = 1,2,3\ldots$ it follows that $f(z_m) = 0$ for $m = 1,2,3\ldots$. But then it follows from the identity theorem for analytic functions that $f(z)$ is identically 0 in D. This is a contradiction of $f(z^*) = c$. We conclude that $\{\phi_n(z)\}$ converges pointwise for all $z \in D$.

Call the limit function $\phi(z)$. We now claim that $\phi_n(z) \rightrightarrows \phi(z)$ in D. This follows from a well known result (see Saks and Zygmund [12, (3.4) on p. 52]) since $\{\phi_n\}$ is a normal family and the sequence converges pointwise to $\phi(z)$. Since, however, a proof is easy, we give one here. If our claim does not hold, there exists a compact set A contained in D, an $\varepsilon > 0$, an increasing sequence of indices $\{n_k\}$, and a sequence of of points $\{z_k\}$ in A such that

$\left|\phi_{n_k}(z_k) - \phi(z_k)\right| \geq \varepsilon$ for $k = 1,2,3\ldots$. Since $\{\phi_n\}$ is a normal family, the sequence $\{\phi_{n_k}\}$ has a subsequence $\{\phi_{n_{k_j}}\}$ which converges uniformly on A to $\phi(z)$.

Thus there exists an integer N such that if $j \geq N$

$|\phi_{n_{k_j}}(z) - \phi(z)| < \varepsilon$ for all $z \in A$. But

$|\phi_{n_{k_j}}(z_{k_j}) - \phi(z_{k_j})| \geq \varepsilon$ for all positive integers j.

This is a contradiction. Therefore $\phi_n(z) \rightrightarrows \phi(z)$ in D.

It now follows from the theorem of Weierstrass appealed to earlier that $\phi(z)$ is analytic in D, and the theorem is proved.

CHAPTER 6

THE "O" TAUBERIAN THEOREM FOR SUMMABILITY (B,α,β)

In this chapter we prove our main Theorem (6.7). For convenience and to emphasize its structure, the proof is divided among three theorems. (For an outline of similar proofs in the case of Borel summability see Hardy [8, section 9.13].)

(6.1) THEOREM.

$$\textit{If } a_n = O(n^{-1/2}) \textit{ and } \sum_{n=0}^{\infty} a_n = A \ (B,\alpha,\beta)$$

then $A_n = O(1)$.

PROOF.

Since $\quad a_n = O(n^{-1/2})$

it follows that $\quad a_n = o(1)$

and thus $\quad A_n = o(n^{1/2}) \quad$ by (1.2) with $\rho = 0$.

Therefore

$$\sum_{n=0}^{\infty} a_n = A \ (e,\alpha/2) \text{ by Theorem (3.8)} .$$

Thus

$$\sqrt{\alpha/2\pi n} \sum_{h=-\infty}^{\infty} e^{-\alpha h^2/2n} A_{n+h} = A + o(1) \quad \text{as } n \to \infty.$$

Since

$$\sqrt{\alpha/2\pi n} \sum_{h=-\infty}^{\infty} e^{-\alpha h^2/2n} = 1 + o(1) \quad \text{as } n \to \infty \quad \text{by (2.4)},$$

it follows that

$$\begin{aligned} A_n \{1 + o(1)\} &= \sqrt{\alpha/2\pi n} \sum_{h=-\infty}^{\infty} e^{-\alpha h^2/2n} A_n \\ &= \sqrt{\alpha/2\pi n} \sum_{h=-\infty}^{\infty} e^{-\alpha h^2/2n} A_{n+h} \\ &\qquad + \sqrt{\alpha/2\pi n} \sum_{h=-\infty}^{\infty} e^{-\alpha h^2/2n} (A_n - A_{n+h}) \\ &= A + o(1) + \sqrt{\alpha/2\pi n} \sum_{|h|>n^\zeta} e^{-\alpha h^2/2n} (A_n - A_{n+h}) \qquad (6.2) \\ &\qquad + \sqrt{\alpha/2\pi n} \sum_{|h|\le n^\zeta} e^{-\alpha h^2/2n} (A_n - A_{n+h}) . \end{aligned}$$

Since $a_n = o(1)$ it follows that $A_n - A_{n+h} = O(|h|)$.

Thus

$$\sqrt{\alpha/2\pi n} \sum_{|h|>n^{\zeta}} e^{-\alpha h^2/2n} (A_n - A_{n+h})$$

$$= O\left\{n^{-1/2} \sum_{|h|>n^{\zeta}} e^{-\alpha h^2/2n} |h|\right\}$$

$$= O\left\{n^{-1/2} \int_0^\infty e^{-\alpha t^2/2n} \, tdt\right\} \qquad \text{by (2.8)}$$

$$= O\left\{n^{-1/2} \left[(n/\alpha) \, e^{-\alpha n^{2\zeta}/2n}\right]\right\}$$

$$= O\left\{n^{1/2} \, e^{-\alpha n^{2\zeta-1}/2}\right\}$$

$= o(1)$ as $n \to \infty$ since $\alpha > 0$ and $2\zeta - 1 > 0$.

That is:

$$\sqrt{\alpha/2\pi n} \sum_{|h|>n^{\zeta}} e^{-\alpha h^2/2n} (A_n - A_{n+h}) = o(1) \quad \text{as } n \to \infty. \tag{6.3}$$

Since $a_n = O(n^{-1/2})$, $|A_{n+h} - A_n| = O\left(\frac{|h|}{\sqrt{n}}\right)$

for $|h| \leq n^{\zeta}$ by (1.14) .

Thus

$$\sqrt{\alpha/2\pi n} \sum_{|h|\leq n^{\zeta}} e^{-\alpha h^2/2n} (A_n - A_{n+h})$$

$$= O\left\{(1/n) \sum_{|h|\leq n^{\zeta}} e^{-\alpha h^2/2n} |h|\right\}$$

$$= O\left\{(1/n) \int_0^{\infty} e^{-\alpha t^2/2n} \, t\,dt\right\} \qquad \text{by (2.7)}$$

$$= O\left\{(1/n) \cdot (n/\alpha)\right\} \qquad \text{by (2.1)}$$

$$= O(1) .$$

That is:

(6.4) $$\sqrt{\alpha/2\pi n} \sum_{|h|\leq n^{\zeta}} e^{-\alpha h^2/2n} (A_n - A_{n+h}) = O(1) .$$

It follows from (6.2), (6.3) and (6.4) that

$$A_n \{1 + o(1)\} = o(1) + O(1) + A + o(1) = O(1) .$$

Therefore $A_n = O(1)$.

(6.5) THEOREM.

If $\sum_{n=o}^{\infty} a_n = A \ (B,\alpha,\beta)$ *and* $A_n = O(1)$ *then*

$\sum_{n=o}^{\infty} a_n = A \ (e,c)$ *for all positive* c.

PROOF.

Let $z = x + iy \in K$ (the complex plane).

Choose $y_o > 0$ and $0 < x_o < \alpha/2$.

Let

$$D = \{z \in K : \quad x > x_o \quad \text{and } |y| < y_o\},$$

$$\phi_n(z) = \sqrt{z/\pi n} \sum_{h=-\infty}^{\infty} e^{-zh^2/n} A_{n+h}$$

$$= \sqrt{z/\pi n} \sum_{h=-n}^{\infty} e^{-zh^2/n} A_{n+h} .$$

and

$$q = q(n) = e^{-x_o/n} .$$

Then $0 < q < 1$ and since $A_n = O(1)$,

$$\left|e^{-zh^2/n} A_{n+h}\right| \leq H\left|e^{-zh^2/n}\right| \qquad \text{for some constant } H$$

$$= H\, e^{-xh^2/n} \leq H \left(e^{-x_o/n}\right)^{h^2} = Hq^{h^2} \quad \text{for all } z \in D.$$

Since $\sum\limits_{h=o}^{\infty} Hq^{h^2} = H \sum\limits_{h=o}^{\infty} q^{h^2} \leq H \sum\limits_{h=o}^{\infty} q^h = \frac{H}{1-q}$

it therefore follows from the Weierstrass M-test that

$\sum\limits_{h=-n}^{\infty} e^{-zh^2/n} A_{n+h}$ converges uniformly in D, and is thus

analytic in D. Since $\sqrt{z/\pi n}$ is analytic in D, $\phi_n(z)$

is analytic in D. $(n = 1,2,3,\ldots .)$

In D,

$$|z| = \sqrt{x^2 + y^2} = x\sqrt{1 + (y/x)^2} \leq x\sqrt{1 + (y_o/x_o)^2}$$

Therefore

$$\sqrt{|z|/n} = O(\sqrt{x/n}) \quad \text{uniformly for all } z \in D.$$

Thus

$$|\phi_n(z)| = O\left((|z|/\pi n) \sum_{h=-\infty}^{\infty} \left|e^{-zh^2/n}\right|\right)$$

$$= O\left(\sqrt{x/\pi n} \sum_{h=-\infty}^{\infty} e^{-xh^2/n}\right)$$

$$= O\{1 + O(\sqrt{x/n})\} \qquad \text{by } (2.3)$$

$$= O(1) \qquad \text{if } x \text{ is bounded above (say if } x_o \leq x \leq x_1).$$

Therefore $\{\phi_n(z)\}$ is almost uniformly bounded in D. Thus the sequence $\{\phi_n\}$ satisfies condition (i) of Theorem (5.1) (Vitali's Theorem).

Since $A_n = O(1)$, $A_n = o(n^{1/2})$

and thus

$$\sum_{n=o}^{\infty} a_n = A\ (e, \alpha/2) \qquad \text{by Theorem (3.8)}.$$

Therefore

(6.6) $$\sum_{n=0}^{\infty} a_n = A\ (e,c) \text{ for } 0 < c \leq \alpha/2, \text{by Theorem (4.22).}$$

Thus

$$\phi_n(c) \to A \quad \text{as } n \to \infty \quad \text{for } x_0 \leq c \leq \alpha/2 \text{ in } D.$$

Thus condition (ii) of Theorem (5.1) is satisfied and it follows from Theorem (5.1) that $\phi_n(z) \to A$ as $n \to \infty$ for all $z \in D$. (Since $\phi(z) = A$ on the closed interval $x_0 \leq x \leq \alpha/2$ it follows from the identity theorem for analytic functions that $\phi(z)$ is identically equal to A in D.)

In particular

$$\sum_{n=0}^{\infty} a_n = A\ (e,c) \quad \text{for } x_0 \leq c < +\infty .$$

Combining this with (6.6) gives

$$\sum_{n=0}^{\infty} a_n = A\ (e,c) \text{ for all } c > 0$$ and the theorem is proved.

The next result is the desired Tauberian theorem.

(6.7) THEOREM.

If $\sum_{n=0}^{\infty} a_n = A\ (B,\alpha,\beta)$ *and* $a_n = O(n^{-1/2})$

then $\sum_{n=0}^{\infty} a_n = A.$

PROOF.

It follows from Theorems (6.1) and (6.5) that

(6.8) $$\sqrt{c/\pi n} \sum_{h=-\infty}^{\infty} e^{-ch^2/n} A_{n+h} = A + o(1) \quad \text{as } n \to \infty$$

for all $c > 0$.

Now since $a_n = o(1)$ it follows that $A_n - A_{n+h} = O(|h|)$,

and

(6.9) $$\sqrt{c/\pi n} \sum_{|h|>n^\zeta} e^{-ch^2/n} (A_n - A_{n+h}) = o(1) \quad \text{as } n \to \infty$$

(exactly as in the proof of (6.3) — just replace $\alpha/2$ there by c here).

Since for any fixed positive c

$$\sqrt{c/\pi n} \sum_{h=-\infty}^{\infty} e^{-ch^2/n} = 1 + o(1) \quad \text{as } n \to \infty \qquad \text{by (2.4)},$$

it follows that

$$A_n \{1 + o(1)\}$$

$$= \sqrt{c/\pi n} \sum_{h=-\infty}^{\infty} e^{-ch^2/n} (A_n - A_{n+h})$$

$$+ \sqrt{c/\pi n} \sum_{h=-\infty}^{\infty} e^{-ch^2/n} A_{n+h}$$

$$= \sqrt{c/\pi n} \sum_{|h|\leq n^\zeta} e^{-ch^2/n} (A_n - A_{n+h}) + o(1) + A + o(1)$$

by (6.8) and (6.9)

Thus

(6.10) $$A_n - A = \sqrt{c/\pi n} \sum_{|h|\leq n^\zeta} e^{-ch^2/n} (A_n - A_{n+h}) + o(1)$$

since $A_n \cdot o(1) = o(1)$ (because $A_n = O(1)$).

For $|h|\leq n^\zeta$, $A_n - A_{n+h} = O\left(\frac{|h|}{\sqrt{n}}\right)$ by (1.14)

and thus

$$\left|A_n - A_{n+h}\right| \leq \frac{H\ |h|}{\sqrt{n}}$$ for some constant H and for

all h with $|h|\leq n^\zeta$ and $n = 1,2,3,\ldots$.

Therefore

$$\left|\sqrt{c/\pi n} \sum_{|h|\leq n^\zeta} e^{-ch^2/n} (A_n - A_{n+h})\right|$$

$$\leq (H/n) \sqrt{c/\pi} \sum_{|h|\leq n^\zeta} e^{-ch^2/n} |h|$$

$$\leq (H/n) \sqrt{c/\pi}\ 2 \left[\int_0^\infty e^{-ct^2/n}\, t\,dt + (n/2ec)^{1/2}\right]$$ by (2.6)

$$= (H/n) \sqrt{c/\pi} \left[2 \int_0^\infty e^{-ct^2/n}\, t\,dt + (2n/ec)^{1/2}\right]$$

$$= (H/n)\sqrt{c/\pi}\left[(n/c) + (2n/ec)^{1/2}\right]$$

$$= H/\sqrt{\pi c} + H\sqrt{2/\pi e}\ n^{-1/2}$$

$$= H/\sqrt{\pi c} + o(1) \quad \text{as} \quad n \to \infty.$$

It follows from (6.10) that

$$|A_n - A| \leq H/\sqrt{\pi c} + o(1) .$$

Thus

$$\limsup_{n\to\infty} |A_n - A| \leq \limsup_{n\to\infty} \left(H/\sqrt{\pi c} + o(1)\right) = H/\sqrt{\pi c} .$$

Since this holds for all positive c, it follows that

$$0 \leq \liminf_{n\to\infty} |A_n - A| \leq \limsup_{n\to\infty} |A_n - A| \leq 0 .$$

Thus

$$\lim_{n\to\infty} |A_n - A| = 0 \quad \text{and } A_n \to A \quad \text{as } n \to \infty.$$

Therefore

$\sum_{n=o}^{\infty} a_n = A$ and the theorem is proved.

(6.11) COROLLARY.

If $a_n = O(n^{-1/2})$ *and either* $\sum_{n=o}^{\infty} a_n = A\ (e,c)$

or $\sum_{n=o}^{\infty} a_n = A\ (\gamma,k)$ *, then* $\sum_{n=o}^{\infty} a_n = A.$

PROOF.

Just combine Theorems (3.1), (3.8), (4.18), and (6.7).

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